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Stochastic extremal problems and the strong Markov property of random fields

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CONTENTS

Introduction

1. This paper introduces the reader to a new section of the theory of Markov fields in which we investigate problems concerned with the notion of the strong Markov property of a random field. This property of a field is formulated in a similar way to the Markov property, but in terms of random rather then deterministic subsets of Euclidean space. It is investigated by means of probabilistic methods and the methods of extremal problems.

In the paper we prove a theorem on the characterization of random domains with respect to which a given field has the strong Markov property. We describe ways of constructing such random domains based on solutions of extremal problems. In the framework of some general models we consider transformations of the type of a "random change of time" which preserve the Markov property. We discuss various examples and applications.

We note that the idea of the strong Markov property is well known for stochastic processes (see [1], [2]). Here it is extended to random fields.

2. Various questions concerning Markov fields have attracted attention for over twenty years (see, for example, [3]–[8]). An important stimulus to the development of this subject was the discovery of its connections with fundamental mathematical problems of quantum theory (see [4], [5]).

At the present moment one apparently cannot distinguish one "standard" definition of Markov fields: for different purposes different versions of this notion are considered. We follow, in general, the approach developed in [3] (although we use slightly different terms). We now formulate the definition of Markov field adopted here, omitting some details; for a completely accurate presentation see §6.3.

We denote by $T(R^d)$ the class of all compact subsets t of Euclidean space R^d that coincide with the closures of their interiors: $t = \text{cl int } t$. We briefly call these sets domains (see [3]). Let a random field, that is, an ordinary or generalized random function, be given on R^d. We say that the given field has the Markov property with respect to some domain $t \subseteq R^d$ if for any $a \subseteq t \subseteq b$, a, $b \in T(R^d)$, the realization of the field on b and its realization on $\widetilde{a}$ ($\widetilde{a} = \text{cl } a^c$—the closure of the complement of a) are conditionally independent given its realization on $\widetilde{a} \cap b$. We call a field Markov if this condition is satisfied for all $t \in T(R^d)$.

We note that the concept of the realization of a field on some closed set $v \subseteq R^d$ includes the values of the field on the set v and on the infinitesimal neighbourhood of v (we have in mind the intersection of the corresponding σ-algebras over all ε-neighbourhoods of v, see §6.1).

In this connection, we also remark that in the particular case $a = t = b$ the above definition leads to a version of the Markov property described in terms of the infinitesimal neighbourhood of the boundary $\partial t = t \cap \widetilde{t} = b \cap \widetilde{a}$—compare [4]-[7].

3. The idea of studying the strong Markov property of random fields arose in the mid-seventies. However, some analogues of this property were used earlier in implicit form in discrete models of statistical physics for the investigation of phase transitions (see, for example, [9] and the references therein). As an independent notion it was introduced in [10], [11] for the study of random fields on a Euclidean space.

The idea of the main definition is as follows. We say that a random field on R^d has the strong Markov property with respect to a random domain(1) $\tau \subseteq R^d$ if for any random domains $\alpha \subseteq \tau \subseteq \beta$ the realization of the field on β and its realization on $\widetilde{\alpha}$ ($= \text{cl } \alpha^c$) are conditionally independent given its realization on $\widetilde{\alpha} \cap \beta$.

The precise formulation will be given later; here we just mention two additional points. Firstly, in the definition given above we should consider only domains α and β depending on τ in a measurable way in some sense. Secondly, information about the behaviour of a field on the random sets β, $\widetilde{\alpha}$, and $\widetilde{\alpha} \cap \beta$ should be furnished with information about the random domains α and β themselves.

(1)By a random domain we mean a measurable map of a probability space into a space of domains $T(R^d)$ equipped with some natural measurable structure.

Among the problems concerned with the strong Markov property the central role is played by the following question. Let us consider a field on R^d that is Markov with respect to some class of deterministic domains. How do we characterize the class of random domains with respect to which the field has the strong Markov property?

The answer to this question is provided by the following result. A Markov field has the strong Markov property with respect to a random domain τ if and only if τ has the following property:

(S) For arbitrary (non-random) domains $a \subseteq b$ and a measurable set D from the space of domains, the event

$$\Delta = \{\tau \in D,\ a \subseteq \tau \subseteq b\}$$

should be representable in the form

$$\Delta = \Gamma_1 \cap \Gamma_2 \pmod 0,$$

where the event Γ_1 is determined by the realization of the field on b, and the event Γ_2 by the realization of the field on the closure of the complement of a (see Theorem 6.1).

Random domains τ satisfying the condition (S) will be called splitting. They are related to splitting times known in the theory of stochastic processes, see [12]-[15].

4. The result presented above reduces the study of the strong Markov property to the study of the condition (S), and so the investigation of splitting random domains becomes the focus of our attention. First of all we would like to learn how to construct such random domains effectively. The key to the answer to this question is the following consideration. It turns out that splitting random domains can be obtained as solutions of some stochastic extremal problems related to the given field. We have in mind extremal problems where a functional and the constraints are random and depend on the realization of the field. Thus, for example, minimizing the integral of the field over a domain (or over its boundary) from a suitable class of domains we obtain as a solution of this minimization problem a splitting random domain.

An analogous fact can be established in a considerably more general context. Namely, for the validity of the results of the kind described above it is necessary, first of all, that the random functional of the domain is consistent with the field and additive (or, more generally, submodular—see §7). The class of domains in which we seek the minimum of the functional should be a lattice[(1)] in the sense of a natural partial ordering of domains: $t \leqslant s \Leftrightarrow t \subseteq s$.

[(1)]A partially ordered set is called a lattice if together with any two elements it contains also their least upper and greatest lower bounds.

A special case of the construction described above is the known construction of splitting time as a minimum point of a stochastic process, see [12]-[15].

Thus, the necessary conditions for an extremum in the problems under consideration acquire a clear probability meaning: extremality implies the splitting condition and strong Markov property. This circumstance not only plays an important role in this paper but also indicates some interesting prospects for further investigations.

5. We also remark that the extremal problems of the kind we investigate here often admit a clear physical interpretation. Generally speaking, they describe optimal processes in random environments. We have in mind the following. It is known that many processes in nature are connected with the formation of random and simultaneously energetically optimal configurations. We can indicate, for example, the formation of cracks in non-homogenous materials, dislocations in crystals, breakdown filaments in dielectrics, and so on. The modelling of such phenomena is a very promising area of application of the theory of stochastic extremal problems. In the present paper we are unable to discuss these examples in a detailed way. Some recent results in this direction will be presented in a separate publication.

6. We note, finally, that it turns out to be convenient to present a considerable part of the exposition in the framework of some general scheme, namely, in the framework of so-called stochastic models on partially ordered sets (see § 1). This scheme includes fields on R^d (the elements of the corresponding partially ordered set are domains), "ordinary" stochastic processes, fields on the space of contours (see [16], [17]), models of the type of [18] connected with stochastic integration on the plane, and a whole range of other models.

Such an approach not only gives the advantage of generality, but also allows us to look into some problems of the theory of random fields and of the theory of stochastic processes from unified positions. This mainly refers to the problem of the construction of Markov processes and fields.

In the theory of Markov processes there are at least two general constructions by means of which from one such process we can construct a whole class of new ones. They are a change of measure by means of an additive functional (see [19]) and a random change of time (first presented in [20]). The construction of Markov fields by means of a change of measure is well known (see, for example, [5], [9]). However, constructions by means of random time change have hardly ever been used in the theory of Markov fields. Stochastic models on partially ordered sets enable us to understand what is the analogue of the random time change in the general situation and to use such constructions to build a whole series of interesting "Markov objects".

7. The paper has the following plan. In §1–5 we present general results related to stochastic models on partially ordered sets. In §6, 7 we pass from abstract stochastic models to random fields on a Euclidean space. In §8 we give a survey of various examples and applications.

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§1. Stochastic models

1.1. Throughout the paper we denote a complete probability space by $(\Omega, \mathcal{F}, P)$.

Let $(T, \leqslant)$ be a partially ordered set, and suppose that to each pair $a, b \in T$, $a \leqslant b$, there correspond three σ-algebras $\mathcal{A}_i(a, b) \subseteq \mathcal{F}$ $(i = 1, 2, 3)$ containing the class $\mathcal{N}(\mathcal{F})$ of all $\mathcal{F}$-measurable sets of measure zero. We say that a family of σ-algebras $\mathfrak{A} = \{\mathcal{A}_i(a, b)\}$ specifies a stochastic model if the following conditions are fulfilled:

1.A. For each $a' \leqslant a \leqslant b \leqslant b'$ we have $\mathcal{A}_i(a, b) \subseteq \mathcal{A}_i(a', b')$ $(i = 1, 2, 3)$.

1.B. $\mathcal{A}_3(a, b) \subseteq \mathcal{A}_1(a, b) \cap \mathcal{A}_2(a, b)$.

A stochastic model $\mathfrak{A}$ is called *Markov* if for any $a \leqslant b$ the σ-algebras $\mathcal{A}_1(a, b)$ and $\mathcal{A}_2(a, b)$ are conditionally independent with respect to $\mathcal{A}_3(a, b)$; symbolically

$$\mathcal{A}_1(a, b) \perp\!\!\!\perp \mathcal{A}_2(a, b) \mid \mathcal{A}_3(a, b).$$

1.2. Let a random element $\xi_t(\omega)$, $\omega \in \Omega$, of a measurable space E_t be given for each $t \in T$, that is, let a random function $\xi_t(\omega)$, $t \in T$, be defined. This random function generates a stochastic model:

$$\begin{aligned} &\mathcal{A}_1(a, b) = \sigma\{\xi_t,\ t \leqslant b\}; \quad \mathcal{A}_2(a, b) = \sigma\{\xi_t,\ t \geqslant a\}; \\ &\mathcal{A}_3(a, b) = \sigma\{\xi_t,\ a \leqslant t \leqslant b\} \quad (a \leqslant b,\ a,\ b \in T), \end{aligned} \tag{1.1}$$

where $\sigma\{\cdot\}$ is the smallest σ-algebra such that all random elements from the family $\{\cdot\}$ and all events from $\mathcal{N}(\mathcal{F})$ are measurable with respect to it. The family of σ-algebras defined above has the property:

$$\mathcal{A}_1(a, b) = \mathcal{A}_1(b, b), \quad \mathcal{A}_2(a, b) = \mathcal{A}_2(a, a). \tag{1.2}$$

Models satisfying (1.2) will be called *regular*.

We say that a random function $\xi_t(\omega)$ is Markov if

$$\sigma\{\xi_p,\ p \leqslant t\} \perp\!\!\!\perp \sigma\{\xi_q,\ q \geqslant t\} \mid \sigma\{\xi_t\},\ t \in T.$$

We note that if the model (1.1) is Markov, then the random function generating it is Markov too. Clearly, the converse is not always true (but it is true if T is linearly ordered). We also note that if $\mathfrak{B} = \{\mathcal{B}_i(a, b)\}$ is a regular Markov model and $\mathcal{B}_3(t, t) = \sigma\{\zeta_t\}$, where ζ_t is a random element for each t, then ζ_t, $t \in T$, is a Markov random function. Thus in a stochastic model to each interval $[a, b] = \{t \in T: a \leqslant t \leqslant b\}$ there correspond three σ-algebras $\mathcal{A}_i(a, b)$ $(i = 1, 2, 3)$ ("past", "future", and "present"). It will be necessary to consider analogous σ-algebras for random intervals.

To this end we assume that T is a partially ordered measurable space, that is, a σ-algebra $\mathcal{T}$ is given on T and $\{(t, s): t \leqslant s\} \in \mathcal{T} \times \mathcal{T}$. Moreover, we suppose that we have a family $\mathcal{H}$ of maps $f_k : T \to T$, $g_k : T \to T$ $(k = 1, 2, ...)$ satisfying the following requirements:(1)

(I) the set of values of $f_k(t)$ and $g_k(t)$ is at most countable $(t \in T,\ k = 1, 2, ...)$;

(II) $f_k(t) \uparrow t$, $g_k(t) \downarrow t$, $t \in T$;

(III) $\mathcal{T}(f_k) \uparrow \mathcal{T}$, $\mathcal{T}(g_k) \uparrow \mathcal{T}$, where $\mathcal{T}(f)$ is the σ-algebra consisting of sets of the form $f^{-1}(\Gamma)$, $\Gamma \in \mathcal{T}$;

(IV) for any $t \leqslant s$ we have the inequalities $f_k(t) \leqslant f_k(s)$, $g_k(t) \leqslant g_k(s)$ $(k = 1, 2, ...)$ (monotonicity of the maps f_k, g_k).

A family of maps $\mathcal{H} = \{f_k, g_k\}$ with the above properties will be called a *skeleton*(2) of a partially ordered measurable space T, and $\{f_k\}$ and $\{g_k\}$ will be called respectively the *left* and *right system of maps* of the skeleton $\mathcal{H}$. A stochastic model $\mathfrak{A} = \{\mathcal{A}_i(a, b)\}$ given on T will be called continuous (with respect to the skeleton $\mathcal{H}$) if for any $a, b \in T$, $a \leqslant b$, the following condition is satisfied:

1.C. As $k \to \infty$, $\mathcal{A}_i(f_k(a), g_k(b)) \downarrow \mathcal{A}_i(a, b)$ $(i = 1, 2, 3)$.

1.3. We note some simple properties of the space T with a skeleton $\mathcal{H}$, which will be repeatedly used in the sequel.

Remark 1.1. We fix any $k = 1, 2, ...$ and we consider the countable set $\mathfrak{S}_k$ of all intervals of the form $[f_k(a), g_k(b)]$, $a, b \in T$, $a \leqslant b$. We number the intervals of $\mathfrak{S}_k$: $\mathfrak{S}_k = \{[p_1, q_1], [p_2, q_2], \ldots\}$. Then the set $\{q_1, q_2, \ldots\}$ is a countable upper bound for T, and $\{p_1, p_2, \ldots\}$ is a countable lower bound for T, that is, for any $t \in T$ there are m and n such that $p_m \leqslant t \leqslant q_n$.

Remark 1.2. The diagonal $D = \{(t, s) \in T \times T : t = s\}$ of the space $T \times T$ is measurable, since

$$D = \{(t, s): t \leqslant s\} \cap \{(t, s): s \leqslant t\} \in \mathcal{T} \times \mathcal{T}.$$

Hence it follows that $\mathcal{T}$ contains one-point sets and any measurable map into $(T, \mathcal{T})$ has a measurable graph.

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(1) If t_k $(k = 1, 2, ...)$ are elements of a partially ordered set T, then the notation $t_k \uparrow t$ $(t_k \downarrow t)$ means that $t_1 \leqslant t_2 \leqslant \ldots$, $t = \sup\{t_k\}$ $(t_1 \geqslant t_2 \geqslant \ldots$, $t = \inf\{t_k\})$. If $\mathcal{T}_k$ $(k = 1, 2, ...)$ and $\mathcal{T}$ are σ-algebras, then we write $\mathcal{T}_k \downarrow \mathcal{T}$ if $\mathcal{T}_1 \supseteq \mathcal{T}_2 \supseteq \ldots$, $\mathcal{T} = \bigcap \mathcal{T}_k$ and $\mathcal{T}_k \uparrow \mathcal{T}$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \ldots$, $\mathcal{T} = \bigvee \mathcal{T}_k$ (that is, $\mathcal{T}$ is generated by all the $\mathcal{T}_k$).

(2) This term was used in [3] and [21] for a different (but similar) notion.

Remark 1.3. There is a function $\varphi : T \to R^1$ that is measurable and strictly monotone, that is, $\varphi(t) < \varphi(s)$ whenever $t \leqslant s,\ t \neq s$. Namely, we consider a countable set

$$W = \{w_1, w_2, \ldots\} = g_1(T) \cup g_2(T) \cup \ldots$$

and we put

$$J(t) = \{j: w_j \geqslant t\}; \quad \varphi(t) = -\sum_{j \in J(t)} 2^{-j}$$

(compare a similar construction in [22]). Then for $t \leqslant s$ we have $J(t) \supseteq J(s)$ and consequently $\varphi(t) \leqslant \varphi(s)$, that is, φ is monotone. If $\varphi(t) = \varphi(s)$, then $J(t) = J(s)$, so $s \leqslant g_k(t)$ $(k = 1, 2, \ldots)$. Now $s \leqslant t$ from (II), and consequently $s = t$.

Remark 1.4. We take an arbitrary (not necessarily continuous) stochastic model $\mathfrak{B} = \{\mathscr{B}_i(a, b)\}$ on T. We suppose that the skeleton $\mathscr{H}$ satisfies the following additional condition:

(V) For any $t \in T$ and $m = 1, 2, \ldots$ there are j, k such that $f_j(f_k(t)) \geqslant f_m(t)$ and $g_j(g_k(t)) \leqslant g_m(t)$.

Putting

$$\check{\mathscr{B}}_i(a, b) = \bigcap_{m=1}^{\infty} \mathscr{B}_i(f_m(a), g_m(b)) \qquad (i = 1, 2, 3),$$

we obtain a new stochastic model $\check{\mathfrak{B}} = \{\check{\mathscr{B}}_i(a, b)\}$, which we call the *closure* of the model $\mathfrak{B}$. This model is continuous, since from (V)

$$\bigcap_{k=1}^{\infty} \check{\mathscr{B}}_i(f_k(a), g_k(b)) = \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \mathscr{B}_i(f_j(f_k(a)), g_j(g_k(b))) =$$

$$= \bigcap_{m=1}^{\infty} \mathscr{B}_i(f_m(a), g_m(b)) = \check{\mathscr{B}}_i(a, b).$$

1.4. Let $\alpha(\omega)$ and $\beta(\omega)$ be two random elements of the space $(T, \mathscr{T})$ such that $\alpha(\omega) \leqslant \beta(\omega)$, $\omega \in \Omega$. Then we call $[\alpha, \beta]$ a *random interval.*

For any $k = 1, 2, \ldots$ and $i = 1, 2, 3$ we consider the smallest σ-algebra $\mathscr{A}_i^k(\alpha, \beta)$ with respect to which $\alpha(\omega)$, $\beta(\omega)$, and all events of the form

$$\Gamma \cap \{f_k(\alpha) = a,\ g_k(\beta) = b\}, \quad \Gamma \in \mathscr{A}_i(a, b), \tag{1.3}$$

where $a, b \in T$, $a \leqslant b$, are measurable. We put

$$\mathscr{A}_i(\alpha, \beta) = \bigcap_{k=1}^{\infty} \mathscr{A}_i^k(\alpha, \beta) \quad (i = 1, 2, 3).$$

Remark 1.5. As k increases the σ-algebras $\mathscr{A}_i^k$ do not grow, and thus

$$\mathscr{A}_i^k(\alpha, \beta) \downarrow \mathscr{A}_i(\alpha, \beta). \tag{1.4}$$

To verify this it is enough to prove that any non-empty set of the form

$$\Delta = \{f_m(\alpha) = a,\ g_m(\beta) = b\} \cap \Gamma$$

belongs to $\mathscr{A}_i^k(\alpha, \beta)$ for any $m \geqslant k$, $a \leqslant b$, and $\Gamma \in \mathscr{A}_i(a, b)$. According to (III) there are measurable maps $f_{k,m}, g_{k,m}: T \to T$ such that

$$f_k(t) = f_{k,m}(f_m(t)), \quad g_k(t) = g_{k,m}(g_m(t)) \tag{1.5}$$

$(m \geqslant k)$. Hence

$$\Delta = \Xi \cap \Xi', \quad \Xi = \{f_m(\alpha) = a, \ g_m(\beta) = b\}, \tag{1.6}$$

$$\Xi' = \{f_k(\alpha) = a', \ g_k(\beta) = b'\} \cap \Gamma, \tag{1.7}$$

where $a' = f_{k,m}(a)$, $b' = g_{k,m}(b)$. Moreover, $a' \leqslant a$, $b' \geqslant b$ by (II), hence $\Gamma \in \mathscr{A}_i(a, b) \subseteq \mathscr{A}_i(a', b')$. Thus, $\Xi, \ \Xi' \in \mathscr{A}_i^k(\alpha, \beta)$, which proves our claim.

Remark 1.6. The above definition of σ-algebras $\mathscr{A}_i(\alpha, \beta)$ $(i = 1, 2, 3)$ (describing "past", "future", and "present" for the random interval $[\alpha, \beta]$) depends, in general, on a previously fixed skeleton $\mathscr{H}$. However, in particular examples it is often possible to establish the invariance of this definition with respect to the choice of $\mathscr{H}$ so long as $\mathscr{H}$ satisfies some supplementary regularity conditions. Namely, this happens for stochastic models related to random fields on R^d (see Lemma 6.3 and Remark 6.2).

Remark 1.7. A whole series of examples refers to models where the set T is finite or countable. In this case we assume throughout that $\mathscr{T}$ consists of all subsets of T and $f_k(t) = g_k(t) \equiv t$.

§2. The strong Markov property. Splitting random elements

2.1. Throughout §2–5 we assume that a continuous stochastic model $\mathfrak{A} = \{\mathscr{A}_i(a, b)\}$ is given on a partially ordered space $(T, \mathscr{T}, \leqslant)$ with a skeleton $\mathscr{H} = \{f_k, g_k\}$.

If $\mathscr{A}_1$ and $\mathscr{A}_2$ are two sets of events, then we denote by $\mathscr{A}_1\mathscr{A}_2$ the set of events that can be represented in the form $\Gamma_1 \cap \Gamma_2$, where $\Gamma_1 \in \mathscr{A}_1$, $\Gamma_2 \in \mathscr{A}_2$.

A random element $\tau(\omega)$ $(\omega \in \Omega)$ will be called *splitting* if the following condition is satisfied:

$(\mathscr{S})$ For all $a \leqslant b$, $a, b \in T$, and any $D \subseteq [a, b]$, $D \in \mathscr{T}$, the event $\{\tau \in D\}$ belongs to $\mathscr{A}_1(a, b)\mathscr{A}_2(a, b)$.

We say that a model $\mathfrak{A}$ has the *strong Markov property* with respect to a random element τ if the following condition holds:

$(\mathscr{S}\mathscr{M})$ For any random elements α and β such that

$$\alpha(\omega) \leqslant \tau(\omega) \leqslant \beta(\omega), \quad \alpha(\omega) = f(\tau(\omega)), \quad \beta(\omega) = g(\tau(\omega))$$

$(f, g : T \to T$ are measurable maps) the σ-algebras $\mathscr{A}_1(\alpha, \beta)$ and $\mathscr{A}_2(\alpha, \beta)$ are conditionally independent with respect to $\mathscr{A}_3(\alpha, \beta)$.

Theorem 2.1. *Let $\mathfrak{A}$ be a Markov model. For it to have the strong Markov property with respect to τ it is sufficient that the random element τ is splitting. This condition is necessary if the following supplementary condition is imposed on the model*:

2.A. *For any $t \in T$ and $k = 1, 2, \ldots$*

$$\mathcal{A}_1(f_k(t),\ g_k(t)) \vee \mathcal{A}_2(f_k(t),\ g_k(t)) = \mathcal{F}.$$

2.2. Theorem 2.1 is proved by means of the following Lemma 2.1, whose proof can be found in [24] (a somewhat more general result was obtained earlier in [23]; compare also [14]).

As always, let $(\Omega,\ \mathcal{F},\ P)$ be a complete probability space. For convenience of presentation, let us agree to call any three σ-algebras $\mathscr{C}_i \subseteq \mathcal{F}$ $(i = 1, 2, 3)$ a *Markov system* if

$$\mathscr{C}_1 \perp\!\!\!\perp \mathscr{C}_2 \mid \mathscr{C}_3, \quad \mathcal{N}(\mathcal{F}) \subseteq \mathscr{C}_3 \subseteq \mathscr{C}_1 \cap \mathscr{C}_2$$

(it is clear that we then have $\mathscr{C}_3 = \mathscr{C}_1 \cap \mathscr{C}_2$).

Let N be a countable set, and let a Markov system of σ-algebras $\mathscr{C}_i(n)$ $(i = 1, 2, 3)$ correspond to each $n \in N$. For a random element $\gamma(\omega) \in N$ $(\omega \in \Omega)$ we denote by $\mathscr{C}_i(\gamma)$ $(i = 1, 2, 3)$ the σ-algebra generated by the events

$$\{\gamma = n\} \cap \Gamma, \quad \Gamma \in \mathscr{C}_i(n), \quad n \in N. \tag{2.1}$$

Lemma 2.1. *For the σ-algebras $\mathscr{C}_i(\gamma)$ $(i = 1, 2, 3)$ to form a Markov system it is sufficient that the random element γ satisfies the condition*

$$\{\gamma = n\} \in \mathscr{C}_1(n)\mathscr{C}_2(n), \quad n \in N. \tag{2.2}$$

This condition is necessary if the following additional requirement is fulfilled:

$$\mathscr{C}_1(n) \vee \mathscr{C}_2(n) = \mathcal{F},\ n \in N. \tag{2.3}$$

2.3. We establish two more auxiliary propositions.

Lemma 2.2. *Let $\lambda = [\alpha, \beta]$ be a random interval in T having a finite or countable set of values L. Then for each $i = 1, 2, 3$ the σ-algebra $\mathcal{A}_i(\lambda) = \mathcal{A}_i(\alpha, \beta)$ consists of events of the form*

$$\bigcup_{l \in L} \{\lambda = l\} \cap \Gamma_l, \quad \Gamma_l \in \mathcal{A}_i(l). \tag{2.4}$$

Proof. 1) We put

$$h_k(l) = [f_k(a),\ g_k(b)], \quad l = [a,\ b]. \tag{2.5}$$

We fix an index $i = 1, 2, 3$ and omit it from now on.

We consider the class $\mathcal{S}^k(\lambda)$ of sets representable in a form like (2.4) but with $\Gamma_l \in \mathcal{A}(h_k(l))$, and we claim that $\mathcal{S}^k(\lambda) = \mathcal{A}^k(\lambda) = \mathcal{A}^k(\alpha, \beta)$. We note, first of all, that the class $\mathcal{S}^k(\lambda)$ is a σ-algebra containing events of the form $\{\lambda = l\} = \{\alpha = a,\ \beta = b\}$. Further, if $\Delta = \{h_k(\lambda) = l'\} \cap \Gamma$, where $\Gamma \in \mathcal{A}(l')$, then

$$\Delta = \bigcup_{l \in L:\ h_k(l) = l'} [\{\lambda = l\} \cap \Gamma] \in \mathcal{S}^k(\lambda),$$

since $\Gamma \in \mathcal{A}(l') = \mathcal{A}(h_k(l))$. Thus, $\mathcal{F}^k(\lambda)$ contains all events generating $\mathcal{A}^k(\lambda)$, so $\mathcal{A}^k(\lambda) \subseteq \mathcal{F}^k(\lambda)$. The reverse inclusion follows from the chain of relations

$$\{\lambda = l\} \cap \Gamma_l = \{\lambda = l\} \cap \{h_k(\lambda) = h_k(l)\} \cap \Gamma \in \mathcal{A}^k(\lambda),$$

where $\Gamma_l \in \mathcal{A}(h_k(l))$.

2) Now let $\Sigma \in \mathcal{A}(\lambda) = \bigcap \mathcal{A}^k(\lambda)$. Then, as we have just shown, for any $k = 1, 2, \ldots$

$$\Sigma = \bigcup_{l \in L} \{\lambda = l\} \cap \Gamma_l^k, \qquad \Gamma_l^k \in \mathcal{A}(h_k(l)).$$

Consequently, $\Sigma = \bigcup[\{\lambda = l\} \cap \Gamma_l]$, where $\Gamma_l = \liminf \Gamma_l^k \in \mathcal{A}(l)$, from 1.C and (2.5). Conversely, if an event Ξ can be represented in the form (2.4), then from 1) $\Xi \in \mathcal{A}^k(\lambda)$ for any k, and consequently $\Xi \in \mathcal{A}(\lambda)$. The lemma is proved.

2.4. ***Lemma*** 2.3. *Let* $\lambda = [\alpha, \beta]$ *be a random interval such that*

$$\{f_k(\alpha) = a, \;\; g_k(\beta) = b\} \in \mathcal{A}_1(a, \;\; b)\mathcal{A}_2(a, \;\; b) \tag{2.6}$$

for all $k = 1, 2, \ldots$ *and* $a, b \in T$, $a \leqslant b$. *If the model is Markov, then*

$$\mathcal{A}_1(\alpha, \;\beta) \perp\!\!\!\perp \mathcal{A}_2(\alpha, \;\beta) \mid \mathcal{A}_3(\alpha, \;\beta). \tag{2.7}$$

Proof (compare [24]). For $i = 1, 2, 3$, $k, m = 1, 2, \ldots, k \leqslant m$, we denote by $\mathcal{A}_i^{k,m}(\lambda)$ the σ-algebra generated by events of the form

$$\{h_k(\lambda) = l', \;\; h_m(\lambda) = l\} \cap \Gamma, \qquad \Gamma \in \mathcal{A}_i(l'),$$

where $l = [a, b]$ and $l' = [a', b']$ are all possible intervals in T. By property (III) of the maps $\{f_k, g_k\}$ (see § 1) we have $\mathcal{A}_i^{k,m}(\lambda) \uparrow \mathcal{A}_i^k(\lambda)$. Moreover, according to (1.4), $\mathcal{A}_i^k(\lambda) \downarrow \mathcal{A}_i(\lambda)$. Thus, since the property of a system of three σ-algebras of being Markov is preserved by the limit passages $\uparrow$ and $\downarrow$, to prove (2.7) it is enough to show that the system $\mathcal{A}_i^{k,m}(\lambda)$ $(i = 1, 2, 3)$ is Markov for any $k \leqslant m$.

To this end we fix $k \leqslant m$ and turn to Lemma 2.1. We denote by γ the pair of random intervals $(h_m(\lambda), h_k(\lambda))$, and by N the set (at most countable) of all values of γ. For $n = (l, l') \in N$ we put $\mathcal{C}_i(n) = \mathcal{A}_i(l')$ $(i = 1, 2, 3)$. Then $\mathcal{A}_i^{k,m}(\lambda) = \mathcal{C}_i(\gamma)$, which follows directly from the definition of the σ-algebras. The condition (2.6) gives $\{\gamma = n\} = \Gamma_1 \cap \Gamma_2 \cap \Gamma_1' \cap \Gamma_2'$, where $\Gamma_i' \in \mathcal{A}_i(l')$, $\Gamma_i \in \mathcal{A}_i(l)$, and since $\{\gamma = n\} \neq \varnothing$, we have $l \subseteq l'$ (since $h_m(\lambda) \subseteq h_k(\lambda)$), and so $\Gamma_i \in \mathcal{A}_i(l')$. Thus, $\{\gamma = n\} \in \mathcal{C}_1(n)\mathcal{C}_2(n)$, and so drawing on Lemma 2.1 we conclude that $\mathcal{C}_i(\gamma) = \mathcal{A}_i^{k,m}(\lambda)$ $(i = 1, 2, 3)$ is a Markov system.

2.5. *Proof of Theorem* 2.1. In view of Lemma 2.3, to prove the sufficiency we need to check only the condition (2.6) for random intervals $[\alpha, \beta] = [f(\tau), g(\tau)]$ described in $(\mathcal{SM})$. If the event indicated in (2.6) occurs, then

$$a = f_k(f(\tau)) \leqslant f(\tau) = \alpha \leqslant \tau \leqslant \beta = g(\tau) \leqslant g_k(g(\tau)) = b,$$

and consequently this event can be represented in the form $\{\tau \in D\}$, where

$$D = \{t \in [a, b]:\ f_k(f(t)) = a,\ \ g_k(g(t)) = b\} \in \mathcal{T}.$$

Thus, (2.6) follows from $(\mathcal{S})$.

To demonstrate the necessity we assume that $(\mathcal{S}\mathcal{M})$ and 2.A hold, we fix any $[a, b]$, $\mathcal{T} \ni D \subseteq [a, b]$, $D \neq \varnothing$, $k \geqslant 1$, and we show that

$$\Delta \equiv \{\tau \in D\} \in \mathcal{A}_1(\bar{a}, \bar{b})\mathcal{A}_2(\bar{a}, \bar{b}), \tag{2.8}$$

where $\bar{a} = f_k(a)$, $\bar{b} = g_k(b)$. Passing to the limit as $k \to \infty$ and employing the continuity of the stochastic model, we obtain the required property: $\Delta \in \mathcal{A}_1(a, b)\mathcal{A}_2(a, b)$. The limit passage is justified for the following reason. If $\mathcal{A}_i^k \downarrow \mathcal{A}_i$ $(i = 1, 2)$ are two sequences of σ-algebras and $\Delta = \Delta_1^k \cap \Delta_2^k \in \mathcal{A}_1^k\mathcal{A}_2^k$ for all k, then $\Delta \in \mathcal{A}_1\mathcal{A}_2$ since $\Delta = \Delta_1 \cap \Delta_2$, where

$$\Delta_i = \bigcup_n \bigcap_{k \geqslant n} \Delta_i^k \in \mathcal{A}_i.$$

Let us verify (2.8). We denote by J the countable set of intervals of the form $[f_m(t), g_m(t)]$ $(m = 1, 2, \ldots, t \in T)$ that are different from $[\bar{a}, \bar{b}]$, and we observe that for any $t \notin D$ there is an interval $[a', b'] \in J$ containing it. For otherwise $f_m(t) = \bar{a}$, $g_m(t) = \bar{b}$ for all m. Hence $\bar{a} = t = \bar{b}$ (since $f_m(t) \uparrow t$ and $g_m(t) \downarrow t$) and so $t \in D$, which is a contradiction. We number the elements of J and taking into account the above remark we define the maps $f, g : T \to T$ by putting $[f(t), g(t)] = [\bar{a}, \bar{b}]$ for $t \in D$, and for $t \notin D$ taking $[f(t), g(t)]$ to be the interval with the least number, belonging to J and containing t. It is clear that f and g are measurable, $f(t) \leqslant t \leqslant g(t)$, and

$$\{\tau \in D\} = \{[f(\tau), g(\tau)] = [\bar{a}, \bar{b}]\}. \tag{2.9}$$

We consider the random interval $[\alpha, \beta] = [f(\tau), g(\tau)]$. By $(\mathcal{S}\mathcal{M})$, $\mathcal{A}_i(\alpha, \beta)$ $(i = 1, 2, 3)$ is a Markov system of σ-algebras, and $\mathcal{A}_i(\alpha, \beta)$ are given by (2.4), because $[\alpha, \beta]$ takes a countable set of values. Moreover, for any value $[p, q]$ of the interval $[\alpha, \beta]$ we can find $\hat{t}$, $t \in T$ and $j = 1, 2, \ldots$ such that

$$[p, q] = [f(t), g(t)] \ni \hat{t}, \quad f(t) \leqslant f_j(\hat{t}), \quad g_j(\hat{t}) \leqslant g(t),$$

and consequently $\mathcal{A}_1(p, q) \vee \mathcal{A}_2(p, q) = \mathcal{F}$ by 2.A. This condition together with (2.4) allows us to apply Lemma 2.1 to $\gamma = [\alpha, \beta]$ and to obtain the relation (2.8) using (2.9).

2.6. *Remark* 2.1. Let $(T, \mathcal{T}, \leqslant)$ be a partially ordered measurable space. let $\{g_k(t)\}_{k=1}^{\infty}$ be a right system of maps, and for each $t \in T$, let $\mathcal{E}_i(t)$ $(i = 1, 2, 3)$ be a Markov system of σ-algebras in $(\Omega, \mathcal{F}, P)$, where

$$\mathcal{E}_i(g_k(t)) \downarrow \mathcal{E}_i(t) \ (k \to \infty) \quad \text{and} \quad \mathcal{E}_i(t) \subseteq \mathcal{E}_i(s)$$

whenever $t \leqslant s$. For a random element τ of the space T we put $\mathscr{E}_i(\tau) = \bigcap_{k=1}^{\infty} \mathscr{E}_i^k(\tau)$, where $\mathscr{E}_i^k(\tau)$ is generated by τ and by the events $\{g_k(\tau) = t\} \cap \Gamma$, $\Gamma \in \mathscr{E}_i(t)$. If for any $t \in T$ and $D \in \mathscr{T}$, $D \subseteq \{s: s \leqslant t\}$ we have $\{\tau \in D\} \in \mathscr{E}_1(t)\mathscr{E}_2(t)$, then $\mathscr{E}_i(\tau)$ is a Markov system of σ-algebras. The converse implication is valid under an additional hypothesis:

$$\mathscr{E}_1(g_k(t)) \vee \mathscr{E}_2(g_k(t)) = \mathscr{F}.$$

This fact generalizes Theorem 2.1 and can be proved similarly. On the other hand, under the condition that in $(T, \mathscr{T}, \leqslant)$ there is a left system of maps $\{f_k(t)\}$, this result is a direct consequence of Theorem 2.1 (one needs to put $\mathscr{A}_i(a, b) = \mathscr{E}_i(b)$).

§3. Extremal problems and splitting random sets

3.1. Our further aim is to describe various effective constructions of splitting random elements. An intermediate step to achieving this aim will be the investigation of a class of random subsets of T, with similar properties to the class of splitting elements.(1)

Let a set $Z(\omega) \subseteq T$ (possibly empty for some ω) correspond to each $\omega \in \Omega$. We say that $Z(\omega)$ is a random set if for any $D \in \mathscr{T}$

$$\{\omega: Z(\omega) \cap D \neq \varnothing\} \in \mathscr{F}. \tag{3.1}$$

We call a random set $Z(\omega)$ splitting if for any $a \leqslant b$ $(a, b \in T)$ and $D \subseteq [a, b]$, $D \in \mathscr{T}$,

$$\{\omega: Z(\omega) \cap D \neq \varnothing\} \in \mathscr{A}_1(a, b)\mathscr{A}_2(a, b). \tag{3.2}$$

We note that (3.2) implies (3.1) (since, by virtue of Remark 1.1, T can be represented as a union of countably many intervals). Moreover, if $Z(\omega)$ consists of one point $\tau(\omega)$ for each ω and if it satisfies (3.2), then $\tau(\omega)$ is a splitting random element.

3.2. Let $F(\omega, t)$ be a real-valued functional defined for $\omega \in \Omega$ and $t \in Z(\omega) \subseteq T$. We denote by $\overline{Z}(\omega)$ the set of all $s \in Z(\omega)$ such that $F(\omega, s) \leqslant F(\omega, t)$, $t \in Z(\omega)$, that is, the set of the points of $Z(\omega)$ where $F(\omega, \cdot)$ attains its minimum.

In this section we give conditions on Z and F under which the following assertion is valid: if $Z(\omega)$ is a splitting random set, then $\overline{Z}(\omega)$ is also a splitting random set. Results of this kind can be used to construct splitting random elements. For let us suppose that we have proved that $\overline{Z}(\omega)$ consists of exactly one point $\tau(\omega)$ for each ω. Then, as we mentioned above, $\tau(\omega)$ is a splitting random element. If $\overline{Z}(\omega)$ contains more than one

(1)In applications to fields on R^d (see §§6, 7) the elements of the space T are domains, and random subsets of T are random classes of domains.

point, then we can repeat the previous operation, considering another functional $G(\omega, t)$ on $\bar{Z}(\omega)$ and the set of its minimum points. Iterating this procedure, we can obtain one-element splitting random sets at some (finite or infinite) step.

In many concrete examples the random set $\bar{Z}(\omega)$ obtained after the first step contains its infimum $\tau(\omega) = \inf \bar{Z}(\omega)$ (understood in the sense of the partial ordering given on T), and the random element $\tau(\omega)$ turns out to be splitting. The splitting property of $\tau(\omega)$ is established by using the fact that the element $\tau(\omega) = \inf \bar{Z}(\omega)$ is the unique minimum point of any strictly monotone functional[(1)] $\Phi(t)$, $t \in T$. Thus, to construct $\tau(\omega)$ we first minimize $F(\omega, t)$ on $Z(\omega)$ and then $\Phi(t)$ on $\bar{Z}(\omega)$, that is, we repeat the procedure twice.

We remark also that in particular examples the set Z or the functional F may not depend on ω, see §8. (A set Z that is independent of ω is a trivial example of a splitting random set.)

3.3. We introduce some definitions and notations which will be necessary in the sequel.

We assume that to each $a, b \in T$, $a \leqslant b$, there correspond two non-empty sets $M_1(a, b), M_2(a, b) \subseteq T$ (left and right zones for the interval $[a, b]$) satisfying the following properties: $(M1)$ if $[a, b] \subseteq [a', b']$ then $M_i(a, b) \subseteq M_i(a', b')$ $(i = 1, 2)$; $(M2)$ $[a, b] \subseteq M_i(a, b)$ $(i = 1, 2)$; $(M3)$ $M_i(a, b) \in \mathscr{T}$ $(i = 1, 2)$; $(M4)$ the set $\{v: v \leqslant t\}$ is contained in the union of $M_1(t, t)$ and $M_2(t, t)$.

The most important example of $M_i(a, b)$ is the following:

$$(3.3) \qquad M_1(a, b) = \{t: t \leqslant b\}, \quad M_2(a, b) = \{t: t \geqslant a\}$$

(regular zones). Whenever we discuss regular models (see §1.2) we assume that $M_i(a, b)$ have the form (3.3). Zones of a different type arise in a natural way in connection with the models described in §8.8.

Let $F(t)$ be a real functional defined on some set $Z \subseteq T$. (We assume here that F and Z do not depend on ω.) We call $F(t)$, $t \in Z$, *sufficient* if the following condition holds:

$(*)$ For any $t \in Z$ the relations

$$(3.4) \qquad \begin{aligned} F(a) &\geqslant F(t), \quad a \in M_1(t, t) \cap Z, \\ F(b) &\geqslant F(t), \quad b \in M_2(t, t) \cap Z, \end{aligned}$$

give $F(s) \geqslant F(t)$, $s \in Z$.

The inequalities (3.4) mean that in the left zone $M_1(t, t)$ and in the right zone $M_2(t, t)$ there are no points $s \in Z$ such that $F(s) < F(t)$. According to $(*)$, the relations (3.4) are sufficient conditions for a minimum of F on Z (it is clear that these conditions are always necessary).

---

(1) The existence of such functionals on T follows from Remark 1.3.

***Proposition* 3.1.** *Let $F(t)$, $t \in Z$, be a sufficient functional, $a \leqslant b$, $a, b \in T$, and $\mathcal{T} \ni D \subseteq [a, b]$. Then the following statements are equivalent:* (a) *$F(\cdot)$ attains its minimum on Z at some point $t \in D$;* (b) *for each $i = 1, 2$ the minimum of $F(\cdot)$ on $M_i(a, b) \cap Z$ is attained at some point $t_i \in D \cap \cap M_i(a, b) \cap Z$.*

Proof. To get (b) from (a) we put $t_i = t$, since $t \in D \subseteq [a, b]$, and from ($M2$) we have $t \in M_i(a, b)$.

Now let (b) hold, that is,

$$F(s) \geqslant F(t_i), \quad s \in Z_i \equiv Z \cap M_i(a, b). \tag{3.5}$$

Since $t_i \in D \subseteq [a, b] \subseteq M_1(a, b) \cap M_2(a, b)$, we have $t_i \in Z_1 \cap Z_2 \cap D$. Hence from (3.5) we obtain

$$F(t_1) = F(t_2). \tag{3.6}$$

We put $t = t_1$. Then for every $s \in Z \cap M_i(t, t)$ we have $F(s) \geqslant F(t)$. Indeed, $s \in Z_i$, since $Z \cap M_i(t, t) \subseteq Z \cap M_i(a, b)$ (see ($M1$)) and from (3.5) and (3.6) we get $F(s) \geqslant F(t_i) = F(t)$. Applying the sufficiency of F we conclude that $t \in D$ is the minimum point of F on Z. The proposition is proved.

3.4. Let a set $Z(\omega) \subseteq T$ correspond to each $\omega \in \Omega$ and let a real functional $F(\omega, t)$ be defined ($\omega \in \Omega$, $t \in Z(\omega)$). We say briefly that this functional is sufficient if for every ω it is sufficient with respect to the variable $t \in Z(\omega)$.

The sufficiency property will play a key role in accomplishing the programme outlined in §3.2. But we will need some more conditions expressing the consistency of $F(\omega, t)$ and $Z(\omega)$ with the given stochastic model $\{\mathcal{A}_i(a, b)\}$.

For $a, b \in T$, $a \leqslant b$, we put

$$F_i^{a,b}(\omega, t, s) = \begin{cases} F(\omega, t) - F(\omega, s), & t, s \in Z(\omega) \cap M_i(a, b), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.7}$$

We call the functional $F(\omega, t)$, $t \in Z(\omega)$, local if[1]

$$F_i^{a,b}(\omega, t, s) \in \mathcal{A}_i(a, b) \times \mathcal{T} \times \mathcal{T} \qquad (a \leqslant b,\ i = 1, 2). \tag{3.8}$$

From (3.8) and (3.7) it follows that

$$\{(\omega, t)\colon t \in Z(\omega) \cap M_i(a, b)\} \in \mathcal{A}_i(a, b) \times \mathcal{T} \tag{3.9}$$

for all $a \leqslant b$ and $i = 1, 2$ (the property that the set $Z(\omega)$ is local).

The conditions (3.8) and (3.9) mean, respectively, that the increments of the functional F and the construction of the set $Z(\omega)$ in the zone $M_i(a, b)$ can be defined using only the information contained in the σ-algebra $\mathcal{A}_i(a, b)$ (see also Proposition 3.2 at the end of this section).

[1] The symbol $\in$ is used here to denote the measurability of functions with respect to σ-algebras.

Now, if $\Gamma \subseteq \Omega$ and $\mathcal{G}$ is a subalgebra of $\mathcal{F}$, then we denote by $\mathcal{G}\,|_\Gamma$ the σ-algebra of all events $\Delta \in \mathcal{F}$ such that

$$\Delta \cap \Gamma = \Delta' \cap \Gamma$$

for some $\Delta' \in \mathcal{G}$. We put

$$\Omega_i^Z(a, b) = \{\omega:\ Z(\omega) \cap M_i(a, b) \neq \varnothing\}$$

and

$$\mathcal{A}_i^Z(a, b) = \mathcal{A}_i(a, b)\,|_{\Omega_i^Z(a, b)}.$$

We say that a functional $F(\omega, t)$, $t \in Z(\omega)$, is weakly local if

$$F_i^{a, b}(\omega, t, s) \in \mathcal{A}_i^Z(a, b) \times \mathcal{T} \times \mathcal{T} \tag{3.10}$$

for all $a \leqslant b$ and $i = 1, 2$. From (3.10) and (3.7) we find that

$$\{(\omega, t):\ t \in Z(\omega) \cap M_i(a, b)\} \in \mathcal{A}_i^Z(a, b) \times \mathcal{T}, \tag{3.11}$$

where $i = 1, 2$, $a, b \in T$, $a \leqslant b$ ($Z(\omega)$ is weakly local).

The condition (3.10) has the following meaning. If it is known that the domain $Z(\omega)$ of a functional $F(\omega, t)$ intersects the zone $M_i(a, b)$, then we can define increments $F(\omega, t) - F(\omega, s)$ on the intersection $Z(\omega) \cap M_i(a, b)$ employing the information contained in $\mathcal{A}_i(a, b)$ $(i = 1, 2)$. (3.11) can be interpreted similarly.

We remark that local sets (or functionals) are weakly local, since $\mathcal{A}_i(a, b) \subseteq \mathcal{A}_i^Z(a, b)$. If the domain Z of a functional F does not depend on ω, then the conditions (3.8) and (3.10) are equivalent, since then $\Omega_i^Z(a, b)$ is empty or coincides with Ω. (See also Proposition 3.2.)

We call a random element $\tau(\omega)$ *local* or *weakly local* if the one-element set $\{\tau(\omega)\}$ has the corresponding property. An equivalent definition will be given in Lemma 5.1.

3.5. Up to the end of §5 we assume that the measurable space $(T, \mathcal{T})$ satisfies the following condition:

3.A. There exist a standard measurable space[1] $(E, \mathcal{E})$ and a measurable map θ: $(E, \mathcal{E}) \to (T, \mathcal{T})$ such that $\theta(E) = T$.

The main result of this section is the following.

***Theorem* 3.1.** *Let $F(\omega, t)$, $t \in Z(\omega)$, be a sufficient weakly local functional and let*

$$\overline{Z}(\omega) = \{t \in Z(\omega):\ F(\omega, t) = \min_{s \in Z(\omega)} F(\omega, s)\} \tag{3.12}$$

be the set of its minimum points on $Z(\omega)$. If $Z(\omega)$ is a splitting random set. then $\overline{Z}(\omega)$ is also a splitting random set.

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[1] A measurable space is called *standard* if it is isomorphic to a Borel subset of a complete separable metric space.

We first establish Theorem 3.2, formulated below, whose first part is used in the proof of Theorem 3.1.

***Theorem* 3.2.** *Let $F(\omega, t)$, $t \in Z(\omega)$, be a weakly local functional. Then the following assertions are true*: 1) *for any $a \leqslant b$, $a, b \in T$ $(i = 1, 2)$,*

$$\{(\omega, t): t \in \bar{Z}_i(\omega, a, b)\} \in \mathcal{A}_i^Z(a, b) \times \mathcal{T}, \tag{3.13}$$

where $\bar{Z}_i(\omega, a, b)$ is the set of minimum points of the functional $F(\omega, t)$ on the set

$$Z_i(\omega, a, b) = Z(\omega) \cap M_i(a, b); \tag{3.14}$$

2) *if $\bar{Z}(\omega)$ is a random set, then $\bar{Z}(\omega)$ is weakly local.*

3.6. The proofs of Theorems 3.1 and 3.2 are based on some general lemmas.

As before, let $(\Omega, \mathcal{F})$ and $(T, \mathcal{T})$ be two measurable spaces, where the first is complete with respect to the measure P and the second satisfies condition 3.A. Let Δ be an arbitrary set from $\mathcal{F} \times \mathcal{T}$.

***Lemma* 3.1.** *The projection $\mathrm{pr}_\Omega \Delta$ is measurable with respect to $\mathcal{F}$.*

***Lemma* 3.2.** *There is a measurable map ξ: $(\Omega, \mathcal{F}) \to (T, \mathcal{T})$ such that $(\omega, \xi(\omega)) \in \Delta$ for all $\omega \in \mathrm{pr}_\Omega \Delta$.*

In the case of standard $(T, \mathcal{T})$ Lemmas 3.1 and 3.2 can be found, for example, in [25], [26]. The general case reduces to this one in the following way. We put

$$\Delta' = \{(\omega, e) \in \Omega \times E: (\omega, \theta(e)) \in \Delta\},$$

where θ: $E \to T$ is the map described in 3.A. It is clear that $\Delta' \in \mathcal{F} \times \mathcal{E}$ and $\mathrm{pr}_\Omega \Delta = \mathrm{pr}_\Omega \Delta'$, since $\theta(E) = T$. Moreover, if $(\omega, \xi'(\omega)) \in \Delta'$ for $\omega \in \mathrm{pr}_\Omega \Delta'$, where ξ': $(\Omega, \mathcal{F}) \to (E, \mathcal{E})$ is a measurable map, then the desired map $\xi(\omega)$ can be defined by the formula $\xi(\omega) = \theta(\xi'(\omega))$.

Now, let $\Phi(\omega, t)$ be a $\mathcal{F} \times \mathcal{T}$-measurable map on $\Omega \times T$ with values in $R^1 \cup \{+\infty\}$.

***Lemma* 3.3.** *The function $\bar{\Phi}(\omega) = \inf_{t \in T} \Phi(\omega, t)$ (taking values in $R^1 \cup \{-\infty\} \cup \{+\infty\}$) is measurable with respect to $\mathcal{F}$.*

This follows immediately from Lemma 3.1, since

$$\{\omega: \bar{\Phi}(\omega) < r\} = \mathrm{pr}_\Omega \{(\omega, t): \Phi(\omega, t) < r\}, \quad r \in R^1.$$

***Lemma* 3.4.** *Let a set $Y(\omega) \subseteq T$ correspond to each $\omega \in \Omega$ and let a real function $H(\omega, t)$ be defined for each $\omega \in \Omega$, $t \in Y(\omega)$. Let the function*

$$H(\omega, t, s) = \begin{cases} H(\omega, t) - H(\omega, s), & t, s \in Y(\omega), \\ +\infty & \textit{otherwise}, \end{cases}$$

be measurable with respect to $\mathcal{F} \times \mathcal{G} \times \mathcal{T}$. Then

$$(3.15)\quad \{(\omega, t)\colon t \in Y(\omega),\quad H(\omega, t) = \min_{s \in Y(\omega)} H(\omega, s)\} \in \mathcal{F} \times \mathcal{T}.$$

Proof. We note first that

$$\Delta \equiv \{(\omega, t)\colon t \in Y(\omega)\} = \{(\omega, t)\colon H(\omega, t, t) < \infty\} \in \mathcal{F} \times \mathcal{T},$$

and consequently by Lemma 3.2 we can find a measurable map $\xi\colon (\Omega, \mathcal{F}) \to (T, \mathcal{T})$ such that $(\omega, \xi(\omega)) \in \Delta$ for $\omega \in \Omega_0 \equiv \{\omega\colon Y(\omega) \neq \varnothing\}$. The function $G(\omega, t) = H(\omega, t, \xi(\omega))$ takes values in $R^1 \cup \{+\infty\}$ and is $\mathcal{F} \times \mathcal{T}$-measurable. Hence by Lemma 3.3 the function $\bar{G}(\omega) = \inf_{t \in T} G(\omega, t)$ is measurable with respect to $\mathcal{F}$. Consequently,

$$(3.16)\qquad \{(\omega, t)\colon +\infty > \bar{G}(\omega) = G(\omega, t)\} \in \mathcal{F} \times \mathcal{T}.$$

We claim now that the sets in (3.15) and (3.16) coincide (we denote them by Q and D respectively). Let $(\omega, t) \in D$. Then $G(\omega, t) < +\infty$, consequently $t \in Y(\omega)$, $\omega \in \Omega_0$, and

$$(3.17)\qquad H(\omega, t) - H(\omega, \xi(\omega)) = G(\omega, t) \leqslant G(\omega, s) = \\ = H(\omega, s) - H(\omega, \xi(\omega)),\quad s \in Y(\omega),$$

hence $H(\omega, t) \leqslant H(\omega, s)$ for $s \in Y(\omega)$, that is, $(\omega, t) \in Q$. Conversely, if $(\omega, t) \in Q$, then $t \in Y(\omega)$, $\omega \in \Omega_0$, and for any $s \in Y(\omega)$

$$+\infty > G(\omega, t) = H(\omega, t) - H(\omega, \xi(\omega)) \leqslant H(\omega, s) - H(\omega, \xi(\omega)) = G(\omega, s).$$

For $s \notin Y(\omega)$ we have $G(\omega, s) = +\infty$ and consequently the inequality $G(\omega, t) \leqslant G(\omega, s)$ holds for all $s \in T$. Hence we conclude that $(\omega, t) \in D$. The lemma is proved.

***Lemma* 3.5.** *Let $\mathcal{G}$ be a σ-subalgebra of $\mathcal{F}$ and let $\Gamma \in \mathcal{F}$. A set A belongs to the σ-algebra $\mathcal{G}\,|_\Gamma \times \mathcal{T}$ if and only if*

$$(3.18)\qquad A = [A' \cap (\Gamma \times T)] \cup [A'' \cap (\Gamma^c \times T)],$$

where $A' \in \mathcal{G} \times \mathcal{T}$, $A'' \in \mathcal{F} \times \mathcal{T}$ and $\Gamma^c = \Omega \setminus \Gamma$.

Proof. From the definition of the σ-algebra $\mathcal{G}\,|_\Gamma$ (see §3.4) it consists of those events Δ for which

$$(3.19)\qquad \Delta = (\Delta' \cap \Gamma) \cup (\Delta'' \cap \Gamma^c),\quad \Delta' \in \mathcal{G},\quad \Delta'' \in \mathcal{F}.$$

We consider the class $\mathcal{H}$ of sets A of the form (3.18). This class is a σ-algebra. It contains all sets $\Delta \times B$, where $B \in \mathcal{T}$ and Δ has the form (3.19), consequently, $\mathcal{H}$ contains $\mathcal{G}\,|_\Gamma \times \mathcal{T}$. Conversely, the classes

$$\mathcal{M}_1 = \{A' \in \mathcal{G} \times \mathcal{T}\colon A' \cap (\Gamma \times T) \in \mathcal{G}\,|_\Gamma \times \mathcal{T}\},$$

$$\mathcal{M}_2 = \{A'' \in \mathcal{F} \times \mathcal{T}\colon A'' \cap (\Gamma^c \times T) \in \mathcal{G}\,|_\Gamma \times \mathcal{T}\}$$

are σ-algebras, and for $\Delta' \in \mathcal{G}$, $\Delta'' \in \mathcal{F}$, $B \in \mathcal{T}$:

$$A' = \Delta' \times B \in \mathcal{M}_1, \quad A'' = \Delta'' \times B \in \mathcal{M}_2.$$

Consequently, $\mathcal{M}_1 \supseteq \mathcal{G} \times \mathcal{T}$, $\mathcal{M}_2 \supseteq \mathcal{F} \times \mathcal{T}$ and so $\mathcal{H} \subseteq \mathcal{G}|_\Gamma \times \mathcal{T}$.

3.7. *Proof of Theorem* 3.2. 1) The relation (3.13) can be obtained by applying Lemma 3.4 to the measurable spaces $(\Omega, \mathcal{A}_i^Z(a, b))$, $(T, \mathcal{T})$, and the functional $H(\omega, t)$, $t \in Y(\omega)$, where $H(\omega, t) = F(\omega, t)$ and $Y(\omega) = Z(\omega) \cap M_i(a, b)$. This lemma is indeed applicable in the situation considered here, since the measurability condition for $H(\omega, t, s)$ follows directly from the definition of the weak local property of $F(\omega, t)$, $t \in Z(\omega)$, and the sets in (3.13) and (3.15) coincide. Moreover, the space $(\Omega, \mathcal{A}_i^Z(a, b))$ is complete with respect to the measure P, since $\mathcal{A}_i^Z(a, b) \supseteq \mathcal{N}(\mathcal{F})$ and the σ-algebra $\mathcal{F}$ is complete.

2) To show that $\bar{Z}(\omega)$ is weakly local we fix $a \leqslant b$ and note that if for some ω the intersection $\bar{Z}(\omega) \cap M_i(a, b)$ is non-empty, then for this ω

$$\bar{Z}(\omega) \cap M_i(a, b) = \bar{Z}_i(\omega, a, b). \tag{3.20}$$

In fact, if $t \in M_i = M_i(a, b)$, $t \in Z(\omega)$, and t minimizes $F = F(\omega, \cdot)$ on $Z = Z(\omega)$, then, of course, t minimizes F on $Z \cap M_i$. Conversely, if we know that $\bar{Z} \cap M_i \neq \emptyset$, that is, F has a minimum on Z at some point $s \in Z \cap M_i$, then any point $t \in Z \cap M_i$ realizing the minimum of F on $Z \cap M_i$ will also realize the minimum of F on Z.

Next, we note that

$$\Delta \equiv \{(\omega, t): t \in \bar{Z}_i(\omega, a, b)\} \subseteq \Omega_i \times T, \tag{3.21}$$

where

$$\Omega_i \equiv \{\omega: Z(\omega) \cap M_i \neq \emptyset\} \supseteq \{\omega: \bar{Z}(\omega) \cap M_i \neq \emptyset\} \equiv \bar{\Omega}_i \in \mathcal{F}. \tag{3.22}$$

As we established in 1), $\Delta \in \mathcal{A}_i^Z(a, b) \times \mathcal{T}$. Consequently, by (3.21) and Lemma 3.5, Δ can be represented in the form

$$\Delta = \Delta' \cap (\Omega_i \times T), \quad \Delta' \in \mathcal{A}_i(a, b) \times \mathcal{T}.$$

Hence from (3.20) and (3.22) we find that

$$\{(\omega, t): t \in \bar{Z}(\omega) \cap M_i\} = \Delta \cap (\bar{\Omega}_i \times T) = \Delta' \cap (\bar{\Omega}_i \times T), \qquad \Delta' \in \mathcal{A}_i(a, b) \times \mathcal{T},$$

which in view of Lemma 3.5 leads us to the desired result:

$$\{(\omega, t): t \in \bar{Z}(\omega) \cap M_i\} \in \mathcal{A}_i^{\bar{Z}}(a, b) \times \mathcal{T}.$$

3.8. *Proof of Theorem* 3.1. We fix $a \leqslant b$ and $\mathcal{T} \ni D \subseteq [a, b]$. Because F is a sufficient functional, from Proposition 3.1 we get

$$A \equiv \{\omega: \bar{Z}(\omega) \cap D \neq \emptyset\} = A_1 \cap A_2,$$

where

$$A_i = \{\omega: \bar{Z}_i(\omega, a, b) \cap D \neq \emptyset\} = \mathrm{pr}_\Omega\{\omega, t): t \in \bar{Z}_i(\omega, a, b) \cap D\} \in \mathcal{A}_i^Z(a, b)$$

as a result of assertion 1) of Theorem 3.2 and Lemma 3.1. This means that $A_i = A_i \cap \Omega_i = A_i' \cap \Omega_i$, where $A_i' \in \mathcal{A}_i(a, b)$ (see (3.22)). Thus,

$$A = A_1 \cap A_2 = A_1 \cap A_2 \cap \{\omega\colon Z(\omega) \cap D \neq \varnothing\} =$$
$$= A_1' \cap A_2' \cap \Omega_1 \cap \Omega_2 \cap \{\omega\colon Z(\omega) \cap D \neq \varnothing) =$$
$$= A_1' \cap A_2' \cap \{\omega\colon Z(\omega) \cap D \neq \varnothing\}.$$

since $Z(\omega) \cap D \neq \varnothing$ implies that $Z(\omega) \cap M \neq \varnothing$ (because $M_i(a, b) \supseteq [a, b] \supseteq D$). It remains to observe that

$$A = A_1' \cap A_2' \cap \{\omega\colon Z(\omega) \cap D \neq \varnothing\} \in \mathcal{A}_1(a, b)\mathcal{A}_2(a, b)$$

because $Z(\omega)$ is a splitting random set.

3.9. To conclude this section, drawing on Lemma 3.1 (which is valid under condition 3.A) we prove the following result.

Proposition 3.2. *Let a set $Z(\omega) \subseteq T$ correspond to each $\omega \in \Omega$. If $Z(\omega)$ is weakly local, then $Z(\omega)$ is a random set. If $Z(\omega)$ is local, then $Z(\omega)$ is a splitting random set.*

Proof. To show that $Z(\omega)$ is a random set it is enough to prove (3.1) for any set $D \in \mathcal{T}$ contained in some interval $[a, b] \subseteq T$. (Each $D \in \mathcal{T}$ is a countable union of such sets.)

Let $\mathcal{T} \in D \subseteq [a, b]$. Then by ($M2$) the set $\Gamma = \{\omega\colon Z(\omega) \cap D \neq \varnothing\}$ can be represented in the form

$$\Gamma = \mathrm{pr}_\Omega \{(\omega, t)\colon t \in Z(\omega) \cap M_1(a, b) \cap D\}.$$

If $Z(\omega)$ is weakly local, then $\Gamma \in \mathcal{A}_1^Z(a, b) \subseteq \mathcal{F}$ from (3.11) and Lemma 3.1. If $Z(\omega)$ is local, then from (3.9) and Lemma 3.1 we get $\Gamma \in \mathcal{A}_1(a, b)$, which necessarily gives (3.2).

§4. Constructions of splitting elements based on the solution of extremal problems

4.1. Drawing on the results of the previous section, we prove some general theorems on the construction of splitting random elements. The constructions presented here are based on arguments contained in §3.2.

Let Z be a $\mathcal{T}$-measurable subset of T, and $F_m(\omega, t)$, $t \in Z$ ($m = 1, 2, \ldots$) a sequence of local functionals with (non-random) domain Z. We assume that for each ω the functionals $\{F_m(\omega, \cdot)\}$ separate points of Z, that is, for any $t, s \in Z$, $t \neq s$, there is an m such that $F_m(\omega, t) \neq F_m(\omega, s)$. Next, we assume that a Hausdorff topology is given on Z, and with respect to it all functionals $F_m(\omega, \cdot)$ are lower semicontinuous and all sets of the form $Z \cap [a, b]$ ($a, b \in T$) are closed.

Let $Z_0(\omega) \subseteq Z$ be a local random set, non-empty and compact for all ω. We define by induction the sequence of sets $Z_0(\omega), Z_1(\omega), Z_2(\omega), \ldots$, denoting by $Z_m(\omega)$ the set of minimum points of $F_m(\omega, t)$ on $Z_{m-1}(\omega)$. We assume that for all m the restriction $H_m(\omega, t)$ of $F_m(\omega, t)$ to $Z_{m-1}(\omega)$ is a sufficient functional.

***Theorem* 4.1.** *For every ω the intersection $Z_0(\omega) \cap Z_1(\omega) \cap \ldots$ consists of one element $\tau(\omega)$, which is weakly local and splitting.*

4.2. The proof of Theorem 4.1 is based on the following lemma.

***Lemma* 4.1.** *Let $Z(\omega)$ be a local set, where for any interval $[a, b]$*

$$(4.1) \qquad \{\omega: Z(\omega) \cap [a, b] \neq \emptyset\} \in \mathcal{A}_1(a, b)\mathcal{A}_2(a, b).$$

Then $Z(\omega)$ is a splitting random set.

Proof. If $\mathcal{T} \ni D \subseteq [a, b]$, then

$$(4.2) \qquad \{\omega: Z(\omega) \cap D \neq \emptyset\} = \{\omega: Z(\omega) \cap [a, b] \neq \emptyset\} \cap B,$$
$$B = \{\omega: Z(\omega) \cap M_1(a, b) \cap D \neq \emptyset\},$$

since $D \subseteq [a, b] \subseteq M_1(a, b)$. Moreover,

$$B = \mathrm{pr}_\Omega\{(\omega, t): t \in Z(\omega) \cap M_1(a, b) \cap D\} \in \mathcal{A}_1^Z(a, b),$$

drawing on the fact that $Z(\omega)$ is weakly local and on Lemma 3.1. Consequently,

$$B = B_1 \cap \{\omega: Z(\omega) \cap M_1(a, b) \neq \emptyset\}, \text{ where } B_1 \in \mathcal{A}_1(a, b).$$

Applying the relation $[a, b] \subseteq M_1(a, b)$, we conclude that in (4.2) B can be replaced by B_1, and together with (4.1) this gives (3.2).

4.3. *Proof of Theorem* 4.1. For each ω the sets $Z_m(\omega)$ $(m = 1, 2, \ldots)$ are non-empty, compact, and form a chain, consequently their intersection is non-empty. It has only one element, since if $t, s \in Z(\omega) \subseteq Z_m(\omega)$, then

$$F_m(\omega, s) = \min\{F_m(\omega, u), u \in Z_{m-1}(\omega)\} = F_m(\omega, t),$$

which means that $t = s$, since the functionals $\{F_m(\omega, \cdot)\}$ separate points.

We prove by induction that $Z_m(\omega)$ is a weakly local splitting random set for any m. For $m = 0$ this is a consequence of Proposition 3.2. Let it be true for $m-1$. Then the functional $H_m(\omega, t) = F_m(\omega, t)$, $t \in Z_{m-1}(\omega)$, is weakly local, since for any $r \in R^1$ and any $[a, b]$

$$\{(\omega, t, s): F_m(\omega, t) - F_m(\omega, s) \leq r,\ t, s \in Z_{m-1}(\omega) \cap M_i(a, b)\} \in$$
$$\in \mathcal{A}_i^{Z_{m-1}}(a, b) \times \mathcal{T} \times \mathcal{T}$$

because $F_m(\omega, t)$, $t \in Z$, is local and $Z_{m-1}(\omega)$ is weakly local. Using assertion 2) of Theorem 3.2 we conclude that $Z_m(\omega)$ is weakly local in view of the fact that $H_m(\omega, t)$, $t \in Z_{m-1}(\omega)$, is sufficient and $Z_m(\omega)$ are splitting random sets by Theorem 3.1.

Now we show that $Z(\omega) = \{\tau(\omega)\}$ is a weakly local splitting random set. We fix $a \leqslant b$, $i = 1, 2$, and note that

$$\{\omega: Z(\omega) \cap M_i(a, b) \neq \emptyset\} \subseteq \{\omega: Z_m(\omega) \cap M_i(a, b) \neq \emptyset\},$$

consequently $\mathcal{A}_i^{Z_m}(a, b) \subseteq \mathcal{A}_i^Z(a, b)$. Therefore, because $Z_m(\omega)$ is weakly local,

$$\Gamma_m \equiv \{(\omega, t)\colon t \in Z_m(\omega) \cap M_i(a, b)\} \in \mathcal{A}_i^Z(a, b) \times \mathcal{T}.$$

Hence

$$\{(\omega, t)\colon t \in Z(\omega) \cap M_i(a, b)\} = \bigcap_{m=1}^{\infty} \Gamma_m \in \mathcal{A}_i^Z(a, b) \times \mathcal{T},$$

that is, $Z(\omega)$ is weakly local. Next, since $Z_m(\omega)$ are compact sets forming a chain and the interval $[a, b] \cap Z$ is closed, we have

$$\{\omega\colon Z(\omega) \cap [a, b] \neq \varnothing\} = \bigcap_{m=1}^{\infty} \{\omega\colon Z_m(\omega) \cap [a, b] \neq \varnothing\} \in \mathcal{A}_1(a, b)\mathcal{A}_2(a, b),$$

where the last relation is valid since $Z_m(\omega)$ are splitting random sets for all m. To complete the proof of the theorem it remains to refer to Lemma 4.1.

4.4. ***Theorem*** 4.2. *Let $Z(\omega)$ be a splitting random set, let $F(\omega, t)$, $t \in Z(\omega)$, be a weakly local sufficient functional, and let $\bar{Z}(\omega)$ be the set of minimum points of $F(\omega, \cdot)$ on $Z(\omega)$. If for each ω the partially ordered set $\bar{Z}(\omega)$ contains its greatest lower bound $\tau(\omega) = \inf \bar{Z}(\omega)$, then $\tau(\omega)$ is a weakly local splitting random element.*

Before we prove this theorem we establish two auxiliary propositions.

Proposition 4.1. *Let Z be a subset of T and $F(t)$, $t \in Z$, a real functional (F and Z are non-random). Let the functional F be monotone, that is, $F(t) \leqslant F(s)$ for $t \leqslant s$, and let the set Z be directed to the left, that is, for any $t, s \in Z$ there is a $v \in Z$ such that $v \leqslant t$ and $v \leqslant s$. Then the functional $F(t)$, $t \in Z$, is sufficient.*

Proof. Suppose that for some $t \in Z$ the inequalities (3.4) hold. We take an arbitrary $s \in Z$ and consider $v \in Z$ such that $v \leqslant t$, $v \leqslant s$. From condition ($M4$) (see §3.3) we get $v \in M_i(t, t)$ for some $i = 1, 2$. Then $F(v) \geqslant F(t)$ because of (3.4), and $F(s) \geqslant F(v) \geqslant F(t)$ because F is monotone, that is, t is a minimum point for F, as required.

Proposition 4.2. *Let $T_0 \in \mathcal{T}$ and let $\varphi(t)$, $t \in T_0$, be a $\mathcal{T}$-measurable functional independent of ω. If $Z(\omega) \subseteq T_0$ is a local (respectively, weakly local) set, then the restriction of $\varphi(t)$ to $Z(\omega)$ is a local (respectively, weakly local) functional.*

Proof. This is true because for any $r \in R^1$, $i = 1, 2$, and $a \leqslant b$ the set

$$\{(\omega, t, s)\colon \varphi(t) - \varphi(s) \leqslant r,\ t, s \in Z(\omega) \cap M_i(a, b)\}$$

belongs to the σ-algebra $\mathcal{A}_i(a, b) \times \mathcal{T} \times \mathcal{T}$ or $\mathcal{A}_i^Z(a, b) \times \mathcal{T} \times \mathcal{T}$, depending on whether $Z(\omega)$ is local or weakly local.

Proof of Theorem 4.2. By Theorem 3.1 $\bar{Z}(\omega)$ is a splitting random set and it is weakly local by Theorem 3.2. Let $\Phi(t)$, $t \in \bar{Z}(\omega)$, be the restriction to $\bar{Z}(\omega)$ of the strictly monotone $\mathscr{T}$-measurable functional $\varphi(t)$, $t \in T$, constructed in Remark 1.3. By Proposition 4.2 the functional $\Phi(t)$, $t \in \bar{Z}(\omega)$, is weakly local and by Proposition 4.1 it is sufficient, since for each ω the set $\bar{Z}(\omega)$, which contains its infimum, is left directed. Thus we can apply Theorems 3.1 and 3.2 to $\Phi(t)$, $t \in \bar{Z}(\omega)$, from which it follows that the set of minimum points of $\Phi(t)$ on $\bar{Z}(\omega)$ is splitting and weakly local. But since $\varphi(t)$ is strictly monotone, this set consists of the unique point $\tau(\omega)$, which proves the theorem.

4.5. We discuss the conditions of Theorem 4.2.

The set $Z(\omega)$ is splitting in each of the following cases: a) Z does not depend on ω; b) $Z(\omega)$ is local; c) $Z(\omega)$ is weakly local and has the property (4.1). The case b) is considered in Proposition 3.2, and c) in Lemma 4.1.

We now list some conditions under which (for any ω) the functional $F(\omega, t)$, $t \in Z(\omega)$, is sufficient and the set $\bar{Z}(\omega)$ contains its infimum.

***Proposition* 4.3.** *Let $M_i(a, b)$ be regular zones, that is, given by the relation* (3.3). *Let Z be a subset of T and $F(t)$ a functional on Z satisfying the following conditions*:

1) *for any $t, s \in Z$ there exist an infimum $t \wedge s \in Z$ and a supremum $t \vee s \in Z$ (the partially ordered set Z is a lattice)*;

2) *for all $t, s \in Z$ the inequality*

$$F(t \wedge s) + F(t \vee s) \leqslant F(t) + F(s) \tag{4.3}$$

holds (the functional F is submodular, see [9]).

Then the functional $F(t)$, $t \in Z$, is sufficient and the set $\bar{Z}$ of its minimum points is closed under the operations $\wedge$ and $\vee$ ($\bar{Z}$ is a sublattice of the lattice Z). If, moreover, F is lower semicontinuous with respect to some Hausdorff topology on Z, Z is compact, and all the sets $\{s \in Z: s \leqslant t\}$, $t \in Z$, are closed, then $\bar{Z}$ is non-empty and contains its infimum.

Proof. We consider $t \in Z$ such that (3.4) holds and we take any $s \in Z$. Since $t \wedge s \in \{t' : t' \leqslant t\} = M_1(t, t)$ and $t \vee s \in \{t' : t' \geqslant t\} = M_2(t, t)$, from (3.4) we have

$$F(t) \leqslant F(t \wedge s), \quad F(t) \leqslant F(t \vee s), \tag{4.4}$$

and so

$$F(s) \geqslant F(t \vee s) + F(t \wedge s) - F(t) \geqslant F(t)$$

by (4.3) and (4.4). Consequently t is a minimum point of F, which means that F is sufficient.

Next, if $u, v \in \bar{Z}$, then $F(u) \leqslant F(u \vee v)$ and $F(v) \leqslant F(u \wedge v)$, but, on the other hand, $F(u) + F(v) \geqslant F(u \wedge v) + F(u \vee v)$. Hence

$$F(u) = F(v) = F(u \wedge v) = F(u \vee v)$$

and consequently $u \vee v \in \bar{Z}$, $u \wedge v \in \bar{Z}$, that is, $\bar{Z}$ is a sublattice of Z.

If F is lower semicontinuous and Z is compact, then $\bar{Z}$ is compact and non-empty. For any $t \in \bar{Z}$ the set $A(t) = \{s \in \bar{Z}\colon s \leqslant t\}$ is compact, and for any $t_1, t_2, \ldots, t_k \in \bar{Z}$ the intersection

$$A(t_1) \cap A(t_2) \cap \ldots \cap A(t_k)$$

is non-empty: it contains $t_1 \wedge t_2 \wedge \ldots \wedge t_k$, since $\bar{Z}$ is a sublattice of Z. Consequently the intersection $A = \cap A(t)$, where t runs through $\bar{Z}$, is also non-empty. It is clear that A consists of only one element $\inf \bar{Z}$.

4.6. We note that the series of conditions we assumed previously to be satisfied for all $\omega \in \Omega$ can be replaced by similar conditions satisfied for almost all ω. Thus, for example, Theorem 4.2 admits the following modification.

Theorem **4.3.** *Let $Z(\omega)$ be a splitting random set and let $F(\omega, t)$, $t \in Z(\omega)$, be a weakly local functional. Suppose that there exists $\Omega^0 \in \mathcal{F}$ such that $P(\Omega^0) = 0$ and for each $\omega \in \Omega \setminus \Omega^0$ the following conditions are satisfied:* a) *the functional $F(\omega, t)$ is sufficient in $t \in Z(\omega)$;* b) *the set of all minimum points of $F(\omega, t)$ on $Z(\omega)$ contains its infimum. If some $\tau(\omega) \in T$ coincides with this infimum almost everywhere, then $\tau(\omega)$ is a weakly local splitting random set.*

Proof. We fix an arbitrary $a_0 \in T$ and we put $Z'(\omega) = \{a_0\}$, $F'(\omega, a_0) = 0$ for $\omega \in \Omega^0$ and $Z'(\omega) = Z(\omega)$, $F'(\omega, t) = F(\omega, t)$ for $\omega \in \Omega^1 \equiv \Omega \setminus \Omega^0$. Since all σ-algebras considered on Ω contain $\mathcal{N}(\mathcal{F})$, then $Z'(\omega)$ is a splitting random set (see (3.2)) and for any $a \leqslant b$, $i = 1, 2$ we have $\mathcal{A}_i^Z(a, b) = \mathcal{A}_i^{Z'}(a, b) \ni \Omega^0, \Omega^1$. Next, the set

$$W(\omega) = \{(t, s)\colon F'(\omega, t) - F'(\omega, s) \leqslant r,\ t, s \in M_i(a, b) \cap Z'(\omega)\}$$

$(r \in R^1)$ coincides for $\omega \in \Omega^1$ with the similar set for the functional $F(\omega, t)$, $t \in Z(\omega)$, and for $\omega \in \Omega^0$ it is either empty or consists of the point (a_0, a_0). Hence $\{(\omega, t, s)\colon (t, s) \in W(\omega)\}$ belongs to $\mathcal{A}_i^{Z'}(a, b) \times \mathcal{T} \times \mathcal{T}$, that is, $F'(\omega, t)$, $t \in Z'(\omega)$, is weakly local. Hence this functional satisfies all the conditions of Theorem 4.2, from which the required assertion follows.

Thus, the formulations "for all ω" and "for almost all ω" are practically equivalent here. This is connected with the fact that in passing from the first to the second it is sufficient to consider at most a countable family of excluded sets. In the next section there appear, generally speaking, uncountable families of sets of measure zero, which require more precise handling.

§5. Random change of variables

5.1. In this section we consider the transformations of Markov models connected with splitting random elements. These transformations are analogous to the random change of time, whose various versions are known in the theory of random processes.

Let a Markov continuous stochastic model $\mathfrak{A} = \{\mathcal{A}_i(a, b)\}$ be given (see §1). Moreover, let us consider a partially ordered set $(S, \leqslant)$ and let a random element $\tau_s(\omega) \in T$ correspond to each $s \in S$. We say that $\{\tau_s\}_{s\in S}$ is a monotone family of random elements if for any $a \leqslant b$ $(a, b \in S)$ the inequality $\tau_a \leqslant \tau_b$ holds (almost surely (a.s.)).

With each pair $\alpha(\omega)$, $\beta(\omega)$ satisfying $\alpha(\omega) \leqslant \beta(\omega)$ a.s. we associate σ-algebras $\mathcal{A}_i(\alpha, \beta)$ $(i = 1, 2, 3)$ by putting $\mathcal{A}_i(\alpha, \beta) = \mathcal{A}_i(\alpha', \beta')$, where α' and β' are arbitrary random elements such that $\alpha'(\omega) \leqslant \beta'(\omega)$ for all ω and $\alpha = \alpha'$, $\beta = \beta'$ a.s. It is clear that the definition of $\mathcal{A}_i(\alpha, \beta)$ does not depend on the choice of variants α' and β' (see §1.4).

***Theorem* 5.1.** *Let $\{\tau_s\}_{s\in S}$ be a monotone family of weakly local splitting random elements. Then the σ-algebras*

$$\mathcal{B}_i(a, b) = \mathcal{A}_i(\tau_a, \tau_b) \quad (i = 1, 2, 3, \ a \leqslant b) \tag{5.1}$$

define a Markov stochastic model $\mathfrak{B}$ on the space $(S, \leqslant)$. If the initial model $\mathfrak{A}$ is regular, then $\mathfrak{B}$ is also regular.

5.2. The proof of Theorem 5.1 will be preceded by some lemmas.

***Lemma* 5.1.** *The following statements are equivalent*:

(a) *$\gamma(\omega)$ is a weakly local random element*;

(b) *for any $D \in \mathcal{T}$, $a \leqslant b$ $(i = 1, 2)$,*

$$\{\omega: \gamma(\omega) \in D \cap M_i(a, b)\} = \Gamma \cap \{\omega: \gamma(\omega) \in M_i(a, b)\} \in \mathcal{F},$$

where Γ is some event from $\mathcal{A}_i(a, b)$.

Proof. Let the condition (b) be fulfilled, which is equivalent to the requirement:

$$\Delta_D \equiv \{\omega: \gamma(\omega) \in D \cap M_i(a, b)\} \in \mathcal{A}_i^{\{\gamma\}}(a, b). \tag{5.2}$$

We fix $a_0 \in T$ and put $\gamma_0(\omega) = \gamma(\omega)$ if $\gamma(\omega) \in M_i(a, b)$ and $\gamma_0(\omega) = a_0$ otherwise. From (5.2) it follows that the map γ_0 is measurable with respect to $\mathcal{A}_i^{\{\gamma\}}(a, b)$, which means that its graph is measurable with respect to $\mathcal{A}_i^{\{\gamma\}}(a, b) \times \mathcal{T}$ (see Remark 1.2). Therefore,

$$\{(\omega, t): t = \gamma(\omega) \in M_i(a, b)\} = \{(\omega, t): t = \gamma_0(\omega), \gamma(\omega) \in M_i(a, b)\} \in$$
$$\in \mathcal{A}_i^{\{\gamma\}}(a, b) \times \mathcal{T},$$

which gives (a). Conversely, from the fact that γ is weakly local and Lemma 3.1 we deduce that

$$\Delta_D = \mathrm{pr}_\Omega\{(\omega, t): t = \gamma(\omega) \in M_i(a, b) \cap D\} \in \mathcal{A}_i^{\{\gamma\}}(a, b).$$

***Lemma* 5.2.** *Let* $[\alpha(\omega), \beta(\omega)]$ *be a random interval and* $\gamma(\omega)$ *a weakly local random element such that* $\gamma(\omega) \in [\alpha(\omega), \beta(\omega)]$. *Then for any* $D \in \mathcal{T}$

$$\{\omega: \gamma(\omega) \in D\} \in \mathcal{A}_1(\alpha, \beta) \cap \mathcal{A}_2(\alpha, \beta). \tag{5.3}$$

Proof. For any $k = 1, 2, \ldots$ the event $\{\gamma \in D\}$ can be represented as a countable union of the events of the form

$$\{\gamma \in D, f_k(\alpha) = p, g_k(\beta) = q\} \quad (p \in f_k(T), q \in g_k(T)).$$

For any $i = 1, 2$ each of them, in turn, admits a representation in the form

$$\Delta \equiv \{\gamma \in D \cap M_i(p, q)\} \cap \{f_k(\alpha) = p, g_k(\beta) = q\}, \tag{5.4}$$

since the relations $f_k(\alpha) = p$, $g_k(\beta) = q$ imply that

$$\gamma \in [\alpha, \beta] \subseteq [p, q] \subseteq M_i(p, q). \tag{5.5}$$

Now by Lemma 5.1 and (5.5)

$$\Delta = \Gamma \cap \{\gamma \in M_i(p, q)\} \cap \{f_k(\alpha) = p, g_k(\beta) = q\} =$$
$$= \Gamma \cap \{f_k(\alpha) = p, g_k(\beta) = q\}, \quad \Gamma \in \mathcal{A}_i(p, q),$$

which means that $\Delta \in \mathcal{A}_i^k(\alpha, \beta)$. Consequently, $\{\gamma \in D\} \in \mathcal{A}_i^k(\alpha, \beta)$ $(k = 1, 2, \ldots)$, therefore, $\{\gamma \in D\} \in \mathcal{A}_i(\alpha, \beta)$ for any $i = 1, 2$, as required.

***Lemma* 5.3.** *Let* $[\alpha(\omega), \beta(\omega)]$ *and* $[\alpha'(\omega), \beta'(\omega)]$ *be two random intervals, where* $[\alpha(\omega), \beta(\omega)] \subseteq [\alpha'(\omega), \beta'(\omega)]$ *and* α, β *are weakly local. Then* $\mathcal{A}_i(\alpha, \beta) \subseteq \mathcal{A}_i(\alpha', \beta')$ $(i = 1, 2)$.

Proof. It is enough to prove that $\mathcal{A}_i^k(\alpha, \beta) \subseteq \mathcal{A}_i^k(\alpha', \beta')(k = 1, 2, \ldots)$. By Lemma 5.2 α and β are measurable with respect to $\mathcal{A}_i^k(\alpha', \beta')$ and we need only show that

$$\Xi \equiv \{f_k(\alpha) = p, g_k(\beta) = q\} \cap \Gamma \in \mathcal{A}_i^k(\alpha', \beta')$$

whenever $\Gamma \in \mathcal{A}_i(p, q)$. We decompose Ξ into a countable union of events

$$\Gamma \cap \{f_k(\alpha) = p, g_k(\beta) = q\} \cap \{f_k(\alpha') = p', g_k(\beta') = q'\}, \tag{5.6}$$

where $p' \in f_k(T)$, $q' \in g_k(T)$. We note that these events are non-empty only when $[p, q] \subseteq [p', q']$, since

$$p' \leqslant f_k(\alpha') \leqslant f_k(\alpha) \leqslant g_k(\beta) \leqslant g_k(\beta') = q'.$$

But if $[p, q] \subseteq [p', q']$, then $\Gamma \in \mathcal{A}_i(p, q) \subseteq \mathcal{A}_i(p', q')$. From this and Lemma 5.2 we conclude that each of the events (5.6) belongs to $\mathcal{A}_i^k(\alpha', \beta')$.

5.3. *Proof of Theorem* 5.1. We note that $\mathcal{A}_3(\gamma, \delta) \subseteq \mathcal{A}_1(\gamma, \delta) \cap \mathcal{A}_2(\gamma, \delta)$ for any random interval $[\gamma, \delta]$ (see (1.3)). Hence it follows that the family of σ-algebras (5.1) satisfies the condition 1.B.

We take $a' \leqslant a \leqslant b \leqslant b'$. Then $\tau_{a'} \leqslant \tau_a \leqslant \tau_b \leqslant \tau_{b'}$ (a.s.), and consequently there are random elements α', α, β, β', such that $\alpha'(\omega) \leqslant \alpha(\omega) \leqslant \beta(\omega) \leqslant \beta'(\omega)$ for all ω, and $\alpha' = \tau_{a'}$, $\alpha = \tau_a$, $\beta = \tau_b$, $\beta' = \tau_{b'}$ (a.s.). As we see from Lemma 5.1, each of the random elements α', α, β, β' is weakly local. Applying Lemma 5.3 to $[\alpha, \beta]$ and $[\alpha', \beta']$ we obtain

$$\mathcal{A}_i(\tau_a, \tau_b) = \mathcal{A}_i(\alpha, \beta) \subseteq \mathcal{A}_i(\alpha', \beta') = \mathcal{A}_i(\tau_{a'}, \tau_{b'}) \quad (i = 1, 2).$$

From this and the equality $\mathcal{A}_3(\tau_a, \tau_b) = \mathcal{A}_1(\tau_a, \tau_b) \cap \mathcal{A}_2(\tau_a, \tau_b)$ we get condition 1.A for the σ-algebras (5.2). This equality is a result of the conditional independence

$$\mathcal{A}_1(\tau_a, \tau_b) \perp\!\!\!\perp \mathcal{A}_2(\tau_a, \tau_b) \mid \mathcal{A}_3(\tau_a, \tau_b),$$

which we prove now.

By Lemma 2.3 and the equality $\mathcal{A}_i(\tau_a, \tau_b) = \mathcal{A}_i(\alpha, \beta)$ $(i = 1, 2, 3)$ it is enough to prove that any event of the form $\Lambda = \{f_k(\alpha) = p, \ g_k(\beta) = q\}$ belongs to $\mathcal{A}_1(p, q)\mathcal{A}_2(p, q)$. Since α and β are splitting, this follows from the relation $\Lambda = \{\alpha \in D_1\} \cap \{\beta \in D_2\}$, where

$$D_1 = \{t \in [p, q]: \ f_k(t) = p\}, \quad D_2 = \{t \in [p, q]: \ g_k(t) = q\}.$$

We assume now that the model $\mathfrak{A}$ is regular. We need to prove the equalities $\mathcal{A}_1(\alpha, \beta) = \mathcal{A}_1(\beta, \beta)$, $\mathcal{A}_2(\alpha, \beta) = \mathcal{A}_2(\alpha, \alpha)$. We verify the first (the second can be verified similarly). It is sufficient to check that the event

$$\Delta = \{\alpha \in D\} \cap \{f_k(\alpha) = a, \ g_k(\beta) = b\} \cap \Gamma \tag{5.7}$$

belongs to the σ-algebra $\mathcal{A}_1^k(\beta, \beta)$ for any $k = 1, 2, \ldots$, $D \in \mathcal{T}$, $a \leqslant b$, and $\Gamma \in \mathcal{A}_1(a, b)$. Namely, events of the form (5.7) and the random element β generate $\mathcal{A}_1^k(\alpha, \beta)$, hence $\mathcal{A}_1^k(\alpha, \beta) \subseteq \mathcal{A}_1^k(\beta, \beta)$ and consequently $\mathcal{A}_1(\alpha, \beta) \subseteq \mathcal{A}_1(\beta, \beta)$. The reverse inclusion holds, by Lemma 5.3. Since $\Delta \subseteq \{\alpha \in [a, b]\} \subseteq \{\alpha \in M_1(a, b)\}$, we have

$$\Delta = \{\alpha \in D \cap M_1(a, b)\} \cap \{f_k(\alpha) = a\} \cap \{g_k(\beta) = b\} \cap \Gamma.$$

Applying Lemma 5.1 to the random element α and to the set $D' = D \cap \{t: \ f_k(t) = a\}$, we find that there is a $\Gamma_1 \in \mathcal{A}_1(a, b)$ for which

$$\Delta = \Gamma_1 \cap \{\alpha \in M_1(a, b)\} \cap \{g_k(\beta) = b\} \cap \Gamma.$$

Here $\{g_k(\beta) = b\} \subseteq \{\alpha \in M_1(a, b)\} = \{\alpha \leqslant b\}$ (see (3.3)), and consequently

$$\Delta = \bigcup_{b \geqslant c \in f_k(T)} \{f_k(\beta) = c, \ g_k(\beta) = b\} \cap \Gamma_1 \cap \Gamma \in \mathcal{A}_1^k(\beta, \beta),$$

since $\Gamma_1 \cap \Gamma \in \mathcal{A}_1(a, b) = \mathcal{A}_1(c, b)$ for any $c \leqslant b$.

5.4. In the theorem given below we can see, in particular, the way of constructing families of random elements $\{\tau_s\}$ described in Theorem 5.1.

***Theorem* 5.2.** *Let the relevant Markov model* $\mathfrak{A}$ *be regular. For every* $s \in S$, *suppose we are given a splitting random set* ${}^sZ(\omega) \subseteq T$ *and a weakly local functional* ${}^sF(\omega, t)$, $t \in {}^sZ(\omega)$, *and suppose that for any* $a, b, s \in S$ $(a \leqslant b)$ *and for almost all* $\omega \in \Omega$ *the following conditions are fulfilled*:

a) *a set* $W(\omega, a, b) \subseteq T$ *can be found such that it is a lattice (in the sense of the partial ordering* $\leqslant$ *given on* T*) and the sets* ${}^aZ(\omega)$ *and* ${}^bZ(\omega)$ *are sublattices of it*;

b) *for any* $u \in {}^aZ(\omega)$, $v \in {}^bZ(\omega)$ *we have* $u \wedge v \in {}^aZ(\omega)$, $u \vee v \in {}^bZ(\omega)$, *and*

$$(5.8) \qquad {}^aF(\omega, u \wedge v) + {}^bF(\omega, u \vee v) \leqslant {}^aF(\omega, u) + {}^bF(\omega, v),$$

where $\wedge$ *and* $\vee$ *are the lattice operations on* $W(\omega, a, b)$;

c) *the set* ${}^s\overline{Z}(\omega)$ *of minimum points of* ${}^sF(\omega, t)$ *on* ${}^sZ(\omega)$ *contains its infimum* $\inf {}^s\overline{Z}(\omega)$. *If, for any* $s \in S$ *and* $\omega \in \Omega$, $\tau_s(\omega)$ *is an element of* T *such that* $\tau_s(\omega) = \inf {}^s\overline{Z}(\omega)$ *almost everywhere, then* $\{\tau_s\}_{s\in S}$ *is a monotone family of weakly local splitting random elements, and the σ-algebras* $\mathcal{A}_i(\tau_a, \tau_b)$ $(a \leqslant b,\ a, b \in S,\ i = 1, 2, 3)$ *give a regular Markov model on the space* $(S, \leqslant)$.

We recall that in regular models the condition of being weakly local is understood with respect to the zones $M_i(a, b)$ defined in (3.3).

Proof of Theorem 5.2. We denote by $\Omega^0(a, b, s) \in \mathcal{N}$ the set on which the hypotheses a)–c) are not fulfilled. From a) and b) it follows that for $s \in S$, $\omega \in \Omega^1 \equiv \Omega \setminus \Omega^0(s, s, s)$ the set ${}^sZ(\omega)$ is a lattice, and the functional ${}^sF(\omega, t)$ is submodular. Therefore, by Proposition 4.3, the functional ${}^sF(\omega, t)$, $t \in {}^sZ(\omega)$, is sufficient in t for $\omega \in \Omega^1$. Together with c) this allows us to use Theorem 4.3, from which we deduce that $\tau_s(\omega)$ is a weakly local splitting random element.

Now suppose that $a \leqslant b$. Then for almost all ω

$$\tau_a \wedge \tau_b \in {}^aZ, \quad \tau_a \vee \tau_b \in {}^bZ,$$
$$0 \leqslant {}^aF(\tau_a \wedge \tau_b) - {}^aF(\tau_a) \leqslant {}^bF(\tau_b) - {}^bF(\tau_a \vee \tau_b) \leqslant 0$$

by b) and since $\tau_a \in {}^a\overline{Z}$, $\tau_b \in {}^b\overline{Z}$. Consequently, ${}^aF(\tau_a \wedge \tau_b) = {}^aF(\tau_a)$ and $\tau_a \wedge \tau_b \in {}^a\overline{Z}$. Hence $\tau_b \geqslant \tau_a \wedge \tau_b \geqslant \tau_a$, since $\tau_a = \inf {}^a\overline{Z}$. Thus, $\tau_b(\omega) \geqslant \tau_a(\omega)$ for almost all ω, that is, we have proved that the family $\{\tau_s\}$ is monotone.

§6. The strong Markov property of random fields on a Euclidean space

6.1. We say that an (ordinary) random field is given on the d-dimensional space R^d if to each $x \in R^d$ there corresponds a random element $\xi_x(\omega)$

($\omega \in \Omega$) of some measurable space E_x. We say that a generalized random field is given on R^d (see, for example, [3], [5]) if to each infinitely differentiable function φ with compact support, $\varphi \in C_0^\infty = C_0^\infty(R^d)$, there corresponds a random element $\xi_\varphi(\omega)$ of a normed linear space V, where $\xi_{a\varphi+b\psi} = a\xi_\varphi + b\xi_\psi$ (a.s.) for all a, $b \in R^1$, φ, $\psi \in C_0^\infty$, and $\xi_{\varphi_k} \to \xi_\varphi$ in probability whenever $\varphi_k \to \varphi$ in C_0^∞.

Let ξ be a random field (ordinary or generalized) and $u \subseteq R^d$ an open set. We denote by $\mathcal{G}^u$ the σ-algebra $\sigma\{\xi_x, x \in u\}$ if $\xi = \xi_x$, $x \in R^d$, is an ordinary field or the σ-algebra $\sigma\{\xi_\varphi, \operatorname{supp} \varphi \Subset u\}$ if $\xi = \xi_\varphi$, $\varphi \in C_0^\infty$, is a generalized field.[1] For any closed $t \subseteq R^d$ we put

$$\mathcal{F}^t = \bigcap_{\varepsilon>0} \mathcal{G}^{t(\varepsilon)}, \tag{6.1}$$

where $t(\varepsilon)$ is the ε-neighbourhood of t. Each σ-algebra $\mathcal{F}^t$ describes the "realization of the field" on the corresponding closed set t (including its infinitesimal neighbourhood).

When considering random fields we will always assume that the following requirement is fulfilled:

$$\mathcal{G}^{R^d} = \mathcal{F}, \tag{6.2}$$

that is, the σ-algebra of all events is generated (mod 0) by the field ξ.

6.2. Everywhere in the sequel we denote by $T = T(R^d)$ the class of all compact sets $t \subseteq R^d$ that coincide with the closures of their interiors. We assume that $\emptyset \in T$. We recall that sets from the class T are called *compact domains* (following the terminology adopted, for instance, in [3] and [27] (vol. I, §8.VIII)) or, for brevity, simply domains.

We denote by $\mathcal{T}$ the σ-algebra of subsets of T generated by the class of sets of the form $\{t \in T: t \Subset u\}$, where u runs through all open subsets of R^d. Random elements of the measurable space $(T, \mathcal{T})$ will be called *random domains.*

6.3. We say that a random field ξ is Markov with respect to the domain $t \in T$ if for any a, $b \in T$, $a \subseteq t \subseteq b$, the σ-algebras $\mathcal{F}^b$ and $\mathcal{F}^{\tilde{a}}$ are conditionally independent with respect to $\mathcal{F}^{\tilde{a}\cap b}$. (As everywhere in this paper, the symbol $\tilde{a}$ denotes cl a^c, the closure of the complement of a.) A field ξ is called *Markov* if this property holds for all domains $t \in T$.

Various classes of random fields which are Markov in the above sense were considered in [3], [8], [23], [24], [28], [29], and other papers. Some equivalent formulations of the Markov property were used there, which we present in §6.11.

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[1] We recall that any σ-algebra of the form $\sigma\{\cdot\}$ (and all σ-subalgebras of $\mathcal{F}$ considered here) contains, by definition, all events of zero probability, and the probability space $(\Omega, \mathcal{F}, P)$ is complete, see §1.

6.4. For $t \in T$ and $\varepsilon > 0$ we denote by $t\,[\varepsilon]$ the closure of the ε-neighbourhood $t(\varepsilon)$ of t. We also put

$$t(-\varepsilon) = \{x \in R^d:\ \rho(x, t^c) > \varepsilon\}$$

and $t\,[-\varepsilon] = \mathrm{cl}\, t(-\varepsilon)$, where ρ is the Euclidean distance. With each $a \subseteq b$, $a, b \in T$, we associate the σ-algebras

$$(6.3) \qquad \mathcal{A}_1(a, b) = \mathcal{F}^b, \quad \mathcal{A}_2(a, b) = \mathcal{F}^{\tilde{a}}, \quad \mathcal{A}_3(a, b) = \mathcal{F}^{\tilde{a} \cap b}.$$

Let α and β be two random domains such that $\alpha(\omega) \subseteq \beta(\omega)$. For any $i = 1, 2, 3$ we consider the σ-algebra $\mathcal{A}_i(\alpha, \beta)$ equal to the intersection over all $\varepsilon > 0$ of the σ-algebras $\mathcal{A}_i^{(\varepsilon)}(\alpha, \beta)$ generated by α, β, and the class of events of the form[1]

$$(6.4) \quad \{\alpha\,[-\varepsilon] \subseteq a\} \cap \{\beta\,[\varepsilon] \supseteq b\} \cap \Gamma, \quad \Gamma \in \mathcal{A}_i(a, b), \quad a, b \in T, \quad a \subseteq b.$$

We call a field ξ *strongly Markov* with respect to the random domain τ if the σ-algebras $\mathcal{A}_1(\alpha, \beta)$ and $\mathcal{A}_2(\alpha, \beta)$ are conditionally independent with respect to $\mathcal{A}_3(\alpha, \beta)$ for all random domains α and β such that $\alpha(\omega) \subseteq \tau(\omega) \subseteq \beta(\omega)$ and $\alpha(\omega) = f(\tau(\omega))$, $\beta(\omega) = g(\tau(\omega))$, where $f, g : T \to T$ are measurable maps.

The σ-algebras $\mathcal{A}_1(\alpha, \beta)$, $\mathcal{A}_2(\alpha, \beta)$, and $\mathcal{A}_3(\alpha, \beta)$ that we have introduced describe the behaviour of the field on β, $\tilde{\alpha}$, and $\tilde{\alpha} \cap \beta$ respectively (including "infinitesimal neighbourhoods" of these sets) and contain, moreover, information about α and β. The above definition of these σ-algebras and the definitions of the strong Markov property related to them is the most convenient formalization for our purposes of the notions presented in part 3 of the introduction.

6.5. A random domain τ will be called *splitting* if for any $a \subseteq b$, $a, b \in T$, and $D \in \mathcal{T}$, $D \subseteq \{t:\ a \subseteq t \subseteq b\}$, the event $\{\tau \in D\}$ can be represented in the form $\Gamma_1 \cap \Gamma_2$, where $\Gamma_i \in \mathcal{A}_i(a, b)$ $(i = 1, 2)$.

***Theorem* 6.1.** *For a Markov field ξ to be strongly Markov with respect to a random domain τ it is necessary and sufficient that the random domain τ is splitting.*

Now from a given field ξ we construct a stochastic model $\mathfrak{A}(\xi)$, and Theorem 6.1 will follow from the corresponding general result for Markov models, namely from Theorem 2.1.

In the space T of domains we introduce a partial ordering by putting $a \leqslant b$ if and only if $a \subseteq b$, and we define the σ-algebras $\mathcal{A}_i(a, b)$ $(i = 1, 2, 3$, $a \leqslant b)$ by (6.3), assuming moreover that $\varnothing \subseteq b$, $b \in T$, and $\mathcal{F}^{\varnothing} = \sigma\{\mathcal{N}(\mathcal{F})\}$, $\mathcal{F}^{R^d} = \mathcal{F}$ (see (6.2)). It is clear that $\mathfrak{A}(\xi) = \{\mathcal{A}_i(a, b)\}$ is a regular stochastic model and that this model is Markov if and only if the field ξ is Markov.

[1]The fact that $\{\alpha[-\varepsilon] \subseteq a\}$ and $\{\beta[\varepsilon] \supseteq b\}$ are actually events will be rigorously proved in §6.8 (see Lemma 6.2).

A *cube of order* k ($k = 1, 2, \ldots$) is a set of the form

$$\{x=(x_1, x_2, \ldots, x_d) \in R^d: \ n_j 2^{-k} \leqslant x_j \leqslant (n_j+1) 2^{-k}\},$$

where n_j ($j = 1, 2, \ldots, d$) are integers. We denote by $f_k(t)$ (respectively, $g_k(t)$) the union of the cubes of order k contained in int t (respectively, intersecting t).

***Theorem* 6.2.** *The family of maps $\mathscr{H} = \{f_k, g_k\}$ that we have constructed is a skeleton of the partially ordered measurable space of domains $(T, \mathscr{T}, \leqslant)$. The model $\mathfrak{A}(\xi)$ is continuous with respect to $\mathscr{H}$. The field ξ is strongly Markov with respect to the random domain τ if and only if the model $\mathfrak{A}(\xi)$ is strongly Markov with respect to τ.*

6.6. We put aside for a while the proof of Theorem 6.2, and derive Theorem 6.1 from it now.

Proof of Theorem 6.1. Directly from the definitions it follows that τ is a splitting random domain if and only if τ is a splitting random element in $\mathfrak{A}(\xi)$. Therefore, to obtain Theorem 6.1 as a consequence of Theorems 2.1 and 6.2 we need only note that condition 2.A is always satisfied for $\mathfrak{A}(\xi)$.

Indeed, $f_k(t) \subseteq \text{int } t \subseteq g_k(t)$ for all $t \in T$, hence the open sets $v = R^d \setminus f_k(t)$ and $w = \text{int } g_k(t)$ give as a union the whole of R^d. Consequently, any function $\varphi \in C_0^\infty$ can be represented in the form $\varphi = \varphi_w + \varphi_v$, where supp $\varphi_v \subseteq v$ and supp $\varphi_w \subseteq w$. Hence $\xi_\varphi = \xi_{\varphi_v} + \xi_{\varphi_w}$ (a.s.), which means that the random variable ξ_φ is measurable with respect to $\mathscr{G}^v \vee \mathscr{G}^w$ (we recall that $\mathscr{N} \subseteq \mathscr{G}^v \vee \mathscr{G}^w$). Comparing this with (6.2), we find that

$$\mathscr{F} = \mathscr{G}^v \vee \mathscr{G}^w \subseteq \mathscr{F}^{-\overline{f_k(t)}} \vee \mathscr{F}^{-g_k(t)} = \mathscr{F},$$

which is equivalent to condition 2.A for $\mathfrak{A}(\xi)$.

6.7. Before proving Theorem 6.2 we establish some lemmas.

***Lemma* 6.1.** *The family $\mathscr{H} = \{f_k, g_k\}$ described in* §6.5 *has the following property*:

6.A. *For any $t \in T$, $\varepsilon > 0$ there is a k such that $f_k(t) \supseteq t[-\varepsilon]$, $t[\varepsilon] \supseteq g_k(t)$, and for any k there is an $\varepsilon' > 0$ such that $t[-\varepsilon] \supseteq f_k(t)$, $t[\varepsilon] \subseteq g_k(t)$ for $0 < \varepsilon < \varepsilon'$.*

Proof. The first two inclusions hold for k such that the diameter of the cube of order k is less than ε. The last two inclusions hold for any sufficiently small $\varepsilon > 0$, since $f_k(t) \subseteq \text{int } t$ and $t \subseteq \text{int } g_k(t)$.

6.8. For the rest of the paper it will be convenient to have some equivalent definitions of the σ-algebra $\mathscr{F}$. We denote by $\mathscr{F}_1, \ldots, \mathscr{F}_{10}$ the σ-algebras of subsets of T generated by the following classes of sets and maps:

($\mathcal{T}_1$) $\{t \in T: t \ni x\}$, $x \in R^d$; ($\mathcal{T}_2$) $\{t \in T: t \subseteq s\}$, $s \in T$;
($\mathcal{T}_3$) $\{t \in T: t \supseteq s\}$, $s \in T$; ($\mathcal{T}_4$) $\{t \in T: t \cap s \neq \varnothing\}$, $s \in T$;
($\mathcal{T}_5$) $t \mapsto \rho(x, t)$, $x \in R^d$; ($\mathcal{T}_6$) $t \mapsto \tau_H(s, t)$, $s \in T \setminus \{\varnothing\}$;
($\mathcal{T}_7$) $t \mapsto f_k(t)$ ($k = 1, 2, \ldots$); ($\mathcal{T}_8$) $t \mapsto g_k(t)$ ($k = 1, 2, \ldots$);
($\mathcal{T}_9$) $t \mapsto t[\varepsilon_k]$ ($k = 1, 2, \ldots$); ($\mathcal{T}_{10}$) $t \mapsto t[-\varepsilon_k]$ ($k = 1, 2, \ldots$).

where ε_k is an arbitrary (but fixed) sequence of positive numbers tending to zero, f_k and g_k are the maps defined in §6.5, and

$$\tau_H(t, s) = \inf\{\varepsilon > 0: t \subseteq s(\varepsilon),\ s \subseteq t(\varepsilon)\},\quad t, s \in T \setminus \{\varnothing\}, \tag{6.5}$$

is the Hausdorff metric.(1) For each $j = 1, 2, \ldots, 10$ we have the following result.

***Lemma* 6.2(j).** $\mathcal{T}_j = \mathcal{T}$.

Proof (compare [25], [30]). Let $B(x, r)$ (respectively, $B[x, r]$) be the open (respectively, closed) ball with radius r and centre at x. The fact that $\mathcal{T}_j \subseteq \mathcal{T}$ follows from the following assertions:

$$(U1)\qquad \{t \not\ni x\} = \bigcup_{k=1}^{\infty} \{t \subseteq R^d \setminus B[x, 1/k]\};$$

$$(U2)\qquad \{t \subseteq s\} = \bigcap_k \{t \subseteq s(1/k)\};$$

$$(U3)\qquad \{t \supseteq s\} = \bigcap_k \{t \ni y_k\};\qquad (U4)\quad \{t \cap s = \varnothing\} = \{t \subseteq s^c\};$$

$$(U5)\qquad \{\rho(x, t) > r\} = \{t \subseteq R^d \setminus B[x, r]\},\quad r > 0;$$

$$(U6)\qquad \{t: \tau_H(t, s) < r\} = \bigcup_{m=2}^{\infty} \bigcap_{k=1}^{\infty} \left\{t: t \subseteq s\left(r - \frac{r}{m}\right),\ \rho(y_k, t) < r - r/m\right\},$$

where $\{y_k\}$ is dense in s;

($U7$) if $a \in f_k(T)$, then $f_k(t) = a$ if and only if $t \supseteq a[1/m]$ for some $m = 1, 2, \ldots$ and $t \not\supseteq b[1/m]$ for any m and any cube b of order k not contained in a (see ($U3$));

($U8$) if $b \in g_k(T)$, then $g_k(t) = b$ if and only if $t \subseteq \operatorname{int} b$ and t intersects all cubes of order k forming b;

$$(U9)\qquad \{t: t[\varepsilon] \subseteq u\} = \bigcup_k \bigcap_m \{t: \rho(x_m, t) > \varepsilon + 1/k\},$$

where $\{x_m\}$ is a dense subset of the complement of the open set u (see ($U5$)):

$$(U10)\qquad \{t: t[-\varepsilon] \subseteq u\} = \bigcup_{m=1}^{\infty} \bigcap_{x \in G(m, u)} \bigcap_{k=1}^{\infty} \{t: B[x, \varepsilon + 1/k] \not\subseteq t\},$$

where $G(m, u)$ is the set of rational points $x \in R^d$ such that $B(x, 1/m) \not\subseteq u$ (see ($U3$)).

(1)We put $\rho(x, \varnothing) = \tau_H(s, \varnothing) = +\infty$ for $x \in R^d$, $s \in T \setminus \{\varnothing\}$.

Next, we note that $\mathcal{T}$ is generated by functions $t \mapsto \rho(x, t)$, $x \in R^d$, since for open u

$$\{t: t \subseteq u\} = \bigcup_k \bigcap_m \{t: \rho(x_m, t) > 1/k\},$$

where $\{x_m\}$ is dense in $R^d \backslash u$. To prove Lemma 6.2(j) it is sufficient to verify now that functions $t \mapsto \rho(x, t)$, $x \in R^d$, are measurable with respect to $\quad$ (j = 1, 2, ..., 10). For j = 6 this follows from the continuity of $\rho(x, t)$, $t \in T \diagdown \{\varnothing\}$, in the metric $\mathfrak{r}_H$, and for the remaining j from the relations

$$\{\rho(t, x) < r\} = \bigcup_m \{t \ni y_m\}, \quad \{y_m\} \subseteq B(x, r) \subseteq \mathrm{cl}\{y_m\};$$

$$\{\rho(t, x) \geqslant r\} = \bigcup_m \{t \subseteq s_m\}, \quad s_m = \{y: r \leqslant \rho(y, x) \leqslant m\};$$

$$\{\rho(t, x) < r\} = \bigcup_m \bigcup_k \{t \supseteq B[y_m, 1/k]\};$$

$$\{\rho(t, x) \leqslant r\} = \{t \cap B[x, r] \neq \varnothing\};$$

$$\rho(x, t) = \inf_k \rho(x, f_k(t)) = \sup_k \rho(x, g_k(t)) = \sup_k \rho(x, t[\varepsilon_k]) = \inf_k \rho(x, t[-\varepsilon_k]).$$

Remark 6.1. From the equality $\mathcal{T} = \mathcal{T}_6$ it follows that the restriction of $\mathcal{T}$ to $T \diagdown \{\varnothing\}$ coincides with the Borel σ-algebra on $T \diagdown \{\varnothing\}$ associated with the metric $\mathfrak{r}_H$.

6.9. *Proof of Theorem* 6.2. We verify the conditions (I)–(IV) formulated in § 1, observing first that $(T, \mathcal{T}, \leqslant)$ is a partially ordered measurable space, since by Lemma 6.2(1)

$$\{(t, s): t \not\subseteq s\} = \bigcup_k [\{(t, s): x_k \in t\} \diagdown \{(t, s): x_k \in s\}] \in \mathcal{T} \times \mathcal{T},$$

where $t, s \in T$ and $\{x_k\}$ is a countable set dense in R^d.

We have $f_k(t) \subseteq t \subseteq g_k(t)$. Moreover, if $s \subseteq g_k(t)$ for all k, then by Lemma 6.1 $s \subseteq t$. Similarly, if $T \ni s \supseteq f_k(t)$ for all k, then $s \supseteq \mathrm{int}\, t$, hence $s \supseteq t$. Thus (II) has been established.

We note now that $f_k(t)$ is the union of the cubes of order k wholly contained in $f_{k+1}(t)$, and $g_k(t)$ is the union of the cubes of order k that contain at least one of the cubes forming $g_{k+1}(t)$. Consequently, $\mathcal{T}(f_k) \subseteq \mathcal{T}(f_{k+1})$, $\mathcal{T}(g_k) \not\subseteq \mathcal{T}(g_{k+1})$. Hence from Lemmas 6.2(7) and 6.2(8) we get (III).

It is clear that (IV) is satisfied and so $\mathcal{H} = \{f_k, g_k\}$ is a skeleton for $(T, \mathcal{T}, \leqslant)$. The continuity of $\mathfrak{A}(\xi)$ with respect to $\mathcal{H}$ (see 1.C) follows from Lemma 6.1 and the relations

$$\mathcal{F}^s = \bigcap_{\varepsilon > 0} \mathcal{F}^{s[\varepsilon]}, \quad \widetilde{t[-\varepsilon]} = \tilde{t}[\varepsilon], \quad t \in T.$$

To prove the last assertion of Theorem 6.2 it is sufficient merely to verify that the definition of σ-algebras $\mathcal{A}_i(\alpha, \beta)$ given in §6.4 for random domains $\alpha \subseteq \beta$ is equivalent to the corresponding definition given in §1.4 in the framework of general stochastic models.

***Lemma* 6.3.** *For any* $\alpha(\omega) \subseteq \beta(\omega)$ *and* $i = 1, 2, 3$ *the σ-algebra* $\mathcal{A}_i(\alpha, \beta)$ *defined in* §1.4 *coincides with the intersection over all* $\varepsilon > 0$ *of the σ-algebras* $\mathcal{A}_i^{(\varepsilon)}(\alpha, \beta)$ *generated by* α, β, *and the events*

$$\{\alpha\,[-\varepsilon] \subseteq a,\ b \subseteq \beta\,[\varepsilon]\} \cap \Gamma, \quad \Gamma \in \mathcal{A}_i(a, b).$$

Proof. We fix $\varepsilon > 0$ and (omitting the index $i = 1, 2, 3$) we show that if $\Delta \in \mathcal{A}^k(\alpha, \beta) \equiv \mathcal{A}^k$ for all k, then $\Delta \in \mathcal{A}^{(\varepsilon)}(\alpha, \beta) \equiv \mathcal{A}^{(\varepsilon)}$. We put

$$\lambda = [\alpha, \beta] = \{t\colon \alpha \subseteq t \subseteq \beta\}, \quad \lambda^{(\varepsilon)} = [\alpha\,[-\varepsilon],\ \beta\,[\varepsilon]], \quad \lambda^k = [f_k(\alpha),\ g_k(\beta)]$$

and denote by L_k the (countable) set of values of λ^k. Then by 6.A

$$\Delta = \bigcup_{k=1}^{\infty} \bigcup_{l \in L_k} \Delta^{k,l}, \quad \Delta^{k,l} = \{\lambda^k \subseteq \lambda^{(\varepsilon)},\ \lambda^k = l\} \cap \Delta,$$

and so it is sufficient to show that $\Delta^{k,l} \in \mathcal{A}^{(\varepsilon)}$ for all $\Delta \in \mathcal{A}^k$. In turn, it is sufficient to establish this for Δ of the form (1.3), but for such Δ if $\Delta^{k,l} \neq \varnothing$, then

$$\Delta^{k,l} = [\{\lambda^{(\varepsilon)} \supseteq [a, b]\} \cap \Gamma] \cap \{\lambda^k \subseteq \lambda^{(\varepsilon)},\ \lambda^k = [a, b]\} \in \mathcal{A}^{(\varepsilon)}$$

since the maps $t \mapsto t\,[\varepsilon]$ and $t \mapsto t\,[-\varepsilon]$ are measurable (see Lemma 6.2).

Now suppose that $\Delta \in \mathcal{A}^{(\varepsilon)}$ for all $\varepsilon > 0$. We take any $k = 1, 2, \ldots$ and claim that $\Delta \in \mathcal{A}^k$. Because of 6.A,

$$\Delta = \bigcup_{n=1}^{\infty} \bigcup_{l \in L_k} \Delta_{n,l}, \quad \Delta_{n,l} = \{\lambda^{(1/n)} \subseteq \lambda^k,\ \lambda^k = l\} \cap \Delta,$$

and consequently it is sufficient to show that $\Delta_{n,l} \in \mathcal{A}^k$ for all $\Delta \in \mathcal{A}^{(1/n)}$. In turn, it is sufficient to prove this for Δ of the form

$$\{\lambda^{(1/n)} \supseteq [a, b]\} \cap \Gamma, \quad \Gamma \in \mathcal{A}(a, b)$$

(see (6.4)), and if Δ has this form, then for $\Delta_{n,l} \neq \varnothing$

$$\Delta_{n,l} = \{\lambda^{(1/n)} \subseteq \lambda^k,\ \lambda^{(1/n)} \supseteq [a, b]\} \cap [\{\lambda^k = l\} \cap \Gamma] \in \mathcal{A}^k,$$

since $[a, b] \subseteq l \equiv [c, d]$, and consequently

$$\Gamma \in \mathcal{A}(a, b) \subseteq \mathcal{A}(c, d).$$

Lemma 6.3 and Theorem 6.2 are proved.

6.10. *Remark* 6.2. The formulation and proof of Lemma 6.3 can be carried over directly to the general case of a continuous stochastic model $\{\mathcal{A}_i(a, b)\}$ with an arbitrary partially ordered measurable space $(T, \mathcal{T}, \leqslant)$, on which besides a skeleton $\mathcal{H} = \{f_k, g_k\}$ we have a family of measurable maps $t \mapsto t\,[-\varepsilon]$ and $t \mapsto t\,[\varepsilon]$, $\varepsilon > 0$, satisfying condition 6.A.

Remark 6.3. As is clear from Lemma 6.3 and the preceding remark, the particular form of the maps f_k and g_k defined in §6.5 is not important in the proof of Theorem 6.2. It is essential only that these maps form a skeleton of the space of domains and that condition 6.A is fulfilled. However, the construction of f_k, g_k given in §6.5 is apparently the most convenient.

Remark 6.4. The results presented here can easily be carried over to the case when ξ is not defined on the whole of R^d but, say, on some open set homeomorphic to R^d. In this case, to define the maps f_k, g_k, instead of cubes of order k we have to take their images under the corresponding homeomorphism. Other generalizations are also possible, employing the technique developed in the first part of this paper. (Compare [3], where fields on metric spaces were considered.)

6.11. We conclude this section with some equivalent formulations of the Markov property of a field (see [3], [8], [23], [24], [28], [29]).

***Proposition* 6.1.** *The following conditions are equivalent*:

(M_1) $\qquad \mathscr{F}^{\tilde{a}} \perp\!\!\!\perp \mathscr{F}^b \mid \mathscr{F}^{\tilde{a}\cap b}$ *for any* $a, b \in T$, $a \subseteq b$;

(M_2) *for any open bounded set* u *and* $\varepsilon > 0$ *we have* $\mathscr{G}^u \perp\!\!\!\perp \mathscr{G}^v \mid \mathscr{G}^{w(\varepsilon)}$, *where* $v = R^d \setminus \mathrm{cl}\, u$ *and* $w = \partial u$;

(M_3) $\mathscr{G}^p \perp\!\!\!\perp \mathscr{G}^q \mid \mathscr{G}^{p\cap q}$ *for any open sets* p *and* q *such that* $p \cup q = R^d$ *and* p *is bounded*;

(M_4) $\mathscr{F}^t \perp\!\!\!\perp \mathscr{F}^s \mid \mathscr{F}^{t\cap s}$ *for any closed sets* t *and* s *such that* $t \cup s = R^d$ *and* t *is bounded.*

Proof. Let (M_1) hold. To establish (M_2) we apply (M_1) to $a = \mathrm{cl}(u \setminus w[\delta])$, $b = u \cup w[\delta]$, where $\delta = \varepsilon - \varepsilon/m$ $(m = 2, 3, \ldots)$. We obtain

$$\mathscr{G}^u \perp\!\!\!\perp \mathscr{G}^v \mid \mathscr{F}^{w[\varepsilon-\varepsilon/m]},$$

because $\tilde{a} = (R^d \setminus u) \cup w[\delta] \supseteq v$, $b \supseteq u$, and $\tilde{a} \cap b = w[\delta]$. Hence (M_2) follows, since

$$\mathscr{F}^{w[\varepsilon-\varepsilon/m]} \uparrow \mathscr{G}^{w(\varepsilon)}.$$

To deduce (M_3) from (M_2) we put

$$q^c = R^d \setminus q, \quad u = (q^c)(\varkappa), \quad v = R^d \setminus \mathrm{cl}\, u, \quad w(\varepsilon) = (\partial u)(\varepsilon),$$

taking $\varkappa > 0$ and $\varepsilon > 0$ such that $\mathrm{cl}\, u \subseteq p$, $w(\varepsilon) \subseteq p \cap q$. Applying the relation $\mathscr{G}^{s_1 \cup s_2} = \mathscr{G}^{s_1} \vee \mathscr{G}^{s_2}$, which is valid for any open sets s_1, s_2, and general properties of conditionally independent σ-algebras (see, for example, [3], Ch. 2, §1), we obtain from (M_2):

$$\mathscr{G}^u \perp\!\!\!\perp \mathscr{G}^v \mid \mathscr{G}^{w(\varepsilon)\cup(v\cap p)\cup(u\cap q)} = \mathscr{G}^{p\cap q},$$

where $\mathcal{G}^u$ can be replaced by $\mathcal{G}^u \vee \mathcal{G}^{p\cap q} = \mathcal{G}^p$ and $\mathcal{G}^v$ by $\mathcal{G}^v \vee \mathcal{G}^{p\cap q} = \mathcal{G}^q$, which gives (M_3).

To deduce (M_4) from (M_3) it is enough to proceed to the limit as $\varepsilon = k^{-1} \to 0\,(k = 1, 2, \dots)$ in the relation

$$\mathcal{G}^{t(\varepsilon)} \perp\!\!\!\perp \mathcal{G}^{s(\varepsilon)} \mid \mathcal{G}^{t(\varepsilon)\cap s(\varepsilon)} = \mathcal{G}^{(t\cap s)(\varepsilon)},$$

which is valid by (M_3).

Finally, (M_1) follows from (M_4) when $t = b$, $s = \widetilde{a}$.

We note that for generalized fields the conditions (M_1)–(M_4) lead to a concept of Markov property different from the one in [6], [7], [31] (for the corresponding discussion see, for example, [3], [8]).

§7. Constructions of splitting domains

7.1. In the previous section it was shown that the study of the strong Markov property and splitting random domains for a field ξ on R^d can be realized in the framework of the stochastic model $\mathfrak{A}(\xi)$. This gives the possibility of applying the general results of §4, particularly Theorem 4.2, to the construction of splitting random domains. However, to apply Theorem 4.2 directly we need to verify that the space of domains $(T, \mathcal{T})$ satisfies condition 3.A. This fact follows from assertion (a) of the following lemma.

Lemma 7.1. (a) *The space* $(T, \mathcal{T})$ *of domains is standard.* (b) *There is a sequence of measurable maps* y_j $(j = 1, 2, \dots)$ *from* $(T, \mathcal{T})$ *to* R^d, *equipped with the Borel σ-algebra* $\mathcal{B}(R^d)$, *such that* $\partial t = \mathrm{cl}\,\{y_j(t)\}$ *for any* $t \in T\diagdown\{\varnothing\}$.

Proof. We consider the space K of all non-empty compact subsets of R^d with the Hausdorff metric (6.5). It is known that K is a complete separable space (see, for example, [32]) and that there is a sequence of measurable maps w_j: $(K, \mathcal{B}(K)) \to (R^d, \mathcal{B}(R^d))$ $(j = 1, 2, \dots)$ such that $\partial k = \mathrm{cl}\,\{w_j(k)\}$ for every $k \in K$. The last result follows, for example, from Theorem 4.2 of [25], which can be applied after verifying that $K_u \equiv \{k\colon \partial k \cap u \neq \varnothing\} \in \mathcal{B}(K)$ for any open ball $u \subseteq R^d$. But this is true, since

$$K_u = \{k\colon k \cap u \neq \varnothing\}\diagdown\{k\colon k \supseteq u\},$$

where the first set is open and the second is closed in K.

Next, we note that the set $K^0 \equiv \{k \in K\colon \operatorname{int} k \neq \varnothing\}$ coincides with the union of closed sets $\{k \in K\colon k \supseteq u_l\}$ $(l = 1, 2, \dots)$, where u_l are balls with rational centres and radii, and consequently belong to $\mathcal{B}(K)$. Let $w'_j(k)$ be the centre of the ball u_m with the smallest number $m \geqslant j$, contained in $k \in K^0$. Then

$$T\diagdown\{\varnothing\} = \bigcap_{i=1}^{\infty} \{k \in K^0\colon \inf_j \mid w_i(k) - w'_j(k) \mid = 0\} \in \mathcal{B}(K),$$

hence (a) follows by Remark 6.1. To construct $y_j(t)$ it is enough to fix $q \in R^d$ and put $y_j(t) = w_j(t)$ for $t \in T \setminus \{\varnothing\}$ and $y_j(\varnothing) = q$.

7.2. Thus, Theorem 4.2 is valid in the model $\mathfrak{A}(\xi)$. We establish a sufficiently general corollary of it.

Let ξ be a random field on R^d (ordinary or generalized), $Z(\omega)$ a class of non-empty domains given for each $\omega \in \Omega$, and $F(\omega, t)$ a real functional defined for all $\omega \in \Omega$ and $t \in Z(\omega)$. As always, we assume that the space T of domains is partially ordered by the inclusion relation $\subseteq$.

Theorem 7.1. *Let the functional $F(\omega, t)$, $t \in Z(\omega)$ be local. Let the following conditions hold for each $\omega \in \Omega$:*

1) *the set $Z(\omega)$ of domains is a lattice and the functional $F(\omega, t)$, $t \in Z(\omega)$, is submodular, that is, $F(\omega, t \vee s) + F(\omega, t \wedge s) \leqslant F(\omega, t) + F(\omega, s)$, $t, s \in Z(\omega)$, where $\vee$ and $\wedge$ are the lattice operations in $Z(\omega)$;*

2) *the set of the domains $t \in Z(\omega)$ that minimize $F(\omega, t)$ on $Z(\omega)$ is non-empty and contains the (unique) domain $\tau(\omega)$ that is smallest with respect to inclusion.*

Then $\tau(\omega)$ is a weakly local splitting random domain.

We note that since the model $\mathfrak{A}(\xi)$ is regular, in accordance with the convention of §3.3 the condition of being local in this model is understood with respect to regular zones (3.3). Consequently, the property of the functional $F(\omega, t)$, $t \in Z(\omega)$, of being local means here that for any $a, b \in T$, $a \subseteq b$, and $r \in (-\infty, +\infty)$ the following relations are satisfied:

$$\{(\omega, t, s)\colon t, s \in Z(\omega), \quad t, s \subseteq b, \quad F(\omega, t) - F(\omega, s) \leqslant r\} \in \\ \in \mathcal{A}_1(a, b) \times \mathcal{T} \times \mathcal{T},$$

$$\{(\omega, t, s)\colon t, s \in Z(\omega), \quad t, s \supseteq a, \quad F(\omega, t) - F(\omega, s) \leqslant r\} \in \\ \in \mathcal{A}_2(a, b) \times \mathcal{T} \times \mathcal{T},$$

where $\mathcal{A}_i(a, b)$ are the σ-algebras generated by ξ (see (6.3)).

Proof of Theorem 7.1. Since the functional $F(\omega, t)$, $t \in Z(\omega)$, is local, its domain $Z(\omega)$ is also local—see §3.4. Consequently, by Proposition 3.2 the random set $Z(\omega)$ is splitting. Next, since the zones $M_i(a, b)$ are regular, from condition 1) and Proposition 4.3 it follows that the functional $F(\omega, t)$, $t \in Z(\omega)$, is sufficient. Thus, the assertion of Theorem 7.1 follows from Theorem 4.2.

Proposition 7.1. *For the conditions of Theorem* 7.1 *to be satisfied it is sufficient that the following requirements are fulfilled:*

(a) *the set $Z(\omega)$ does not depend on ω $[Z(\omega) = Z]$, it is compact in the Hausdorff metric, and it is a lattice;*

(b) *the functional $F(\omega, t)$ is continuous in $t \in Z$ with respect to the Hausdorff metric and is submodular;*

(c) *for any $t, s \in Z$, $t \subseteq s$, the random variable $F(\omega, t) - F(\omega, s)$ is measurable with respect to $\mathcal{A}_3(t, s)$.*

Proof. Condition 2) follows from Proposition 4.3, so we need only verify that $F(\omega, t)$, $t \in Z(\omega)$, is local. We fix $a \subseteq b$, $a, b \in t$, and put $M_1 = \{t\colon t \subseteq b\}$, $M_2 = \{t\colon t \supseteq a\}$. We note first that for any $i = 1, 2$

$$F(\omega, t) - F(\omega, s) \in \mathcal{A}_i(a, b) \tag{7.1}$$

whenever $t, s \in M_i \cap Z$. For example, if $i = 2$, then

$$t \vee s \in M_i \cap Z, \quad t \subseteq t \vee s, \quad s \subseteq t \vee s,$$

$$F(t) - F(s) = [F(t) - F(t \vee s)] + [F(t \vee s) - F(s)] \in$$
$$\in \mathcal{A}_3(t, t \vee s) \vee \mathcal{A}_3(s, t \vee s) \subseteq \mathcal{A}_i(a, b).$$

From (7.1) and the fact that the function $F(\omega, t) - F(\omega, s)$ is continuous in (t, s) on the metric compact set $(Z \cap M_i) \times (Z \cap M_i)$ we deduce that this function is measurable in (ω, t, s) with respect to the product of $\mathcal{A}_i(a, b)$ and the Borel σ-algebra of the above compact set. Taking account of the fact that $Z \cap M_i \in \mathcal{T}$ and the Borel subsets of $Z \cap M_i$ coincide with $\mathcal{T}$-measurable subsets (see Remark 6.1), we see that f is local, as required.

Remark 7.1. One of the simplest cases when the conditions (a)–(c) given in Proposition 7.1 can be directly verified is the following: (i) Z is a compact (in the Hausdorff metric r_H) class of domains that is closed under finite unions and intersections and is such that $\operatorname{mes}(t \Delta s) \leqslant C \cdot r_H(t, s)$, $t, s \in Z$, where $\operatorname{mes}(\cdot)$ is the Lebesgue measure and C is some constant; (ii) F is defined by $F(\omega, t) = \int_t \eta_x(\omega)dx$, where $\eta_x(\omega)$ is a random function, continuous in $x \in R^d$, such that for all $t \in T$, $x \in t$ the random variable η_x is measurable with respect to $\mathcal{F}^t$ ($\mathcal{F}^t$ are the σ-algebras connected with the field ξ, see §6.1). An example of a set Z satisfying condition (i) is given in §8.1.

§8. Survey of examples and applications

8.1. In this section we present a survey of analytic examples illustrating the results on splitting random domains obtained above. We also consider the applications of the random change of variables described in §5 to the construction of Markov models on the space of contours.

Limitations on the length of this paper do not allow us to analyse the examples in detail, with full proofs. For this purpose we would need a range of tools broader than the one employed here, namely: some techniques connected with Hausdorff measures, estimates of ε-entropy of sets in function spaces, some facts concerning random fields (percolation theorems), limit theorems of probability theory, and others. Therefore, the detailed presentation of the results of this section and some of their generalizations will be given separately in forthcoming papers.

We begin with examples related to one of the most important and well studied Markov fields—a free field [7, 33–35].

Let ξ_φ, $\varphi \in C_0^\infty(R^2)$, be a free field on the plane, that is, a stationary Gaussian random function with zero mean and spectral density $(1+x_1^2+x_2^2)^{-1}$, $x_1, x_2 \in R^2$. The random function ξ_φ can be extended in a canonical way to the Sobolev space $\mathscr{H}_{-1}(R^2)$ of generalized functions with moderate growth such that

$$\int\limits_{R^2}\int \frac{|\tilde{\varphi}(x_1, x_2)|^2}{1+x_1^2+x_2^2}\, dx_1\, dx_2 < \infty,$$

where $\sim$ denotes the Fourier transform (see [34], [35]).

The space $\mathscr{H}_{-1}(R^2)$ contains the indicator functions χ_t of all domains $t \in T(R^2)$, so for all $t \in T(R^2)$ the random variable $\Xi_t(\omega) = \xi_{\chi_t}(\omega)$ $(\omega \in \Omega)$ is well defined (up to equivalence).

Let $\mathfrak{X}_-$ and $\mathfrak{X}_+$ be two classes of continuous functions $a(r)$, $r \in [0, 1]$, such that $a(r) > 0$ for $r \in (0, 1)$. Let the following conditions be fulfilled:

(α) Each of the classes $\mathfrak{X}_-$, $\mathfrak{X}_+$ is closed with respect to uniform convergence and the operations of taking the maximum and the minimum of two functions.

(β) All functions $a(\cdot) \in \mathfrak{X}_- \cup \mathfrak{X}_+$ are bounded by some constant M and satisfy a Lipschitz condition with constant M.

To each pair of functions $f = (f_-, f_+) \in \mathfrak{X}_- \times \mathfrak{X}_+$ there corresponds a domain

$$t_f = \{(r, q) \in R^2\colon\ -f_-(r) \leqslant q \leqslant f_+(r),\quad r \in [0, 1]\}.$$

We denote the class of all such domains by $Z = Z(\mathfrak{X}_-, \mathfrak{X}_+)$.

Theorem 8.1. 1) *The set Z of domains is compact in the Hausdorff metric and is a lattice with $t \wedge s = t \cap s$ and $t \vee s = t \cup s$.* 2) *There is a continuous (with respect to the Hausdorff metric) modification $F(\omega, t)$ of the functional $\Xi_t(\omega)$, $t \in Z$, which for all $\omega \in \Omega$ satisfies the following condition*:

$$F(\omega, t \vee s) + F(\omega, t \wedge s) = F(\omega, t) + F(\omega, s),\quad t, s \in Z.$$

3) *For each $\omega \in \Omega$ the functional $F(\omega, t)$, $t \in Z$, attains its minimum $\bar{F}(\omega)$. The set*

$$\bar{Z}(\omega) = \{t \in Z\colon\ F(\omega, t) = \bar{F}(\omega)\}$$

contains the smallest (with respect to inclusion) domain $\tau(\omega)$. The domain $\tau(\omega)$ is splitting.

Remark 8.1. Employing the theorem on uniqueness of the minimum of a Gaussian random function (see [36]), one can show that $\bar{Z}(\omega) = \{\tau(\omega)\}$ for almost all $\omega \in \Omega$.

The construction of a functional $F(\omega, t)$ described in 2) is based on the results of [37], [38] on the continuity of Gaussian random functions on a metric space. In this an essential role is played by the estimate

$$\overline{\lim_{\varepsilon\downarrow 0}} \frac{\log\log N(\varepsilon, Z)}{\log(1/\varepsilon)} \leqslant 1, \tag{8.1}$$

where $N(\varepsilon, Z)$ is the smallest number of elements in an ε-net of Z. The inequality (8.1) can be derived from formulae on the ε-entropy of classes of functions satisfying a Lipschitz condition [39]. The splitting property of τ can be established by means of Theorem 7.1 and Proposition 7.1.

8.2. It is of interest to study limit distributions connected with extremal random domains. In general, this problem is very difficult. Here we show one case in which it can be solved.

We consider the above construction and apply it to the classes $\mathfrak{X}_-$ and $\mathfrak{X}_+$, where $\mathfrak{X}_-$ consists of one function $f_- \equiv c$, and $\mathfrak{X}_+$ consists of constant functions $f_+ \equiv a$, $a \in [0, A]$ (the numbers $c > 0$ and $A > 0$ are assumed to be given). The domains that belong to the class $Z = Z(\mathfrak{X}_-, \mathfrak{X}_+)$ are rectangles of the form $[0, 1] \times [-c, a]$, $a \in [0, A]$.

Let $\tau_A = [0, 1] \times [-c, \sigma_A(\omega)]$ be the smallest (with respect to inclusion) rectangle among those that minimize $F(\omega, t)$ on Z, where $F(\omega, t)$ was constructed in Theorem 8.1. By this theorem τ_A is a splitting random domain for the free field. The result given below allows us to obtain approximate formulae for distributions of such domains for large values of A.

Theorem 8.2. *For any $r \in (0, 1)$*

$$\lim_{A\to\infty} P\{\sigma_A/A < r\} = \frac{2}{\pi} \arcsin \sqrt{r}.$$

Thus, the distributions of the random variables σ_A/A converge weakly to the arcsin law. In the proof of Theorem 8.2 we use Theorem 12.1 from [40].

8.3. We consider one more example of the construction of splitting random domains, this time related to ordinary random fields.

Let J be the set of all non-empty domains $t \subseteq R^2$ whose boundaries are contours (that is, they are homeomorphic to a circle) and have finite length $l(\partial t)$.

Let $\xi_x(\omega)$ ($x \in R^2$, $\omega \in \Omega$) be a real random function with continuous realizations. We fix $t_0 \in J$, $c \in R^1$ and consider the set

$$q(\omega) = \{x \in R^2 \colon \xi_x(\omega) \geqslant c\}$$

and the class of domains

$$W(\omega) = \{t \in J \colon \partial t \subseteq q(\omega),\ t \supseteq t_0\}.$$

We assume that $W(\omega) \neq \varnothing$, $\omega \in \Omega$.

Theorem 8.3. *In the class $W(\omega)$ there are domains t with minimal perimeter $l(\partial t)$. Among the domains with minimal perimeter there is a (unique) domain $\tau(\omega)$ that is smallest with respect to inclusion. It is splitting.*

The boundary $\partial\tau(\omega)$ of $\tau(\omega)$ gives a solution to the following problem: find the shortest path around the domain t_0 on the set $R^2 \backslash \mathrm{int}\ t_0$ not intersecting $\{x\colon \xi_x < c\}$

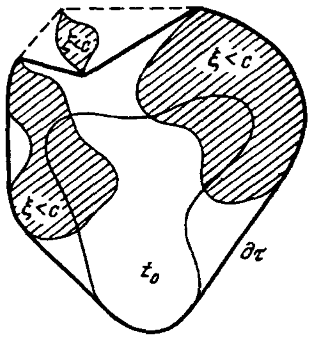

Fig. 1

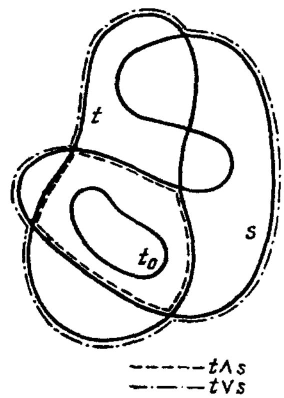

Fig. 2

(the shortest path around the "island" t_0 on the "sea" $R^2 \backslash \mathrm{int}\ t_0$ outside the "shallows" $\{x\colon \xi_x(\omega) < c\}$, see Fig. 1).

The proof of Theorem 8.3 is based on Theorem 7.1 and the following result.

Proposition 8.1. *For each* $\omega \in \Omega$ *the set* $W(\omega)$ *is a lattice, where if* $t, s \in W(\omega)$, *then* $t \wedge s = \mathrm{cl}\, u$, *and* $t \vee s = v^c (= R^2 \setminus v)$, *where* u *is the connected component of* int $t \cap$ int s *containing* int t_0, *and* v *is the unbounded connected component of* $t^c \cap s^c$ *(see Fig. 2). The functional* $l(\partial t)$ *is submodular on* $W(\omega)$.

Despite the fact that this assertion is "geometrically obvious", to prove it rigorously one has to use some subtle results on the topology of the plane ([27], vol. 2, §61). Moreover, during the proof it is necessary to extend the functional $l(\cdot)$ in a suitable way to a class of sets broader than the class of contours. To this end we consider the Hausdorff measure of order 1 on Borel subsets of R^2 and use a number of properties of it [41], [42].

8.4. We distinguish two important special classes of splitting random domains. Let ξ be a field on R^d and $\mathcal{F}^t$, $t \in T = T(R^d)$, the σ-algebras defined by (6.1).

Theorem 8.4. *If a random domain* τ *satisfies one of the conditions*:

$$\{\tau \subseteq t\} \in \mathcal{F}^t, \quad t \in T, \tag{8.2}$$

$$\{\tau \supseteq t\} \in \mathcal{F}^t, \quad t \in T, \tag{8.3}$$

then it is splitting.

Thus, if (8.2) or (8.3) is fulfilled and the given field ξ is Markov, then it has the strong Markov property with respect to τ (compare [3], [10], [11]).

We give a simple proof of Theorem 8.4. Let (8.2) hold. To prove the splitting property it is sufficient to show that

$$\{\tau \in D,\ \tau \subseteq b\} \in \mathscr{F}^b,\quad D \in \mathscr{T},\ b \in T. \tag{8.4}$$

In turn, it is sufficient to prove (8.4) for sets D of the form

$$D = \{t \in T\colon\ t \subseteq c\},\quad c \in T, \tag{8.5}$$

since by Lemma 6.2(2) such sets generate $\mathscr{T}$. But if D has the form (8.5), then by (8.2)

$$\{\tau \in D,\ \tau \subseteq b\} = \{\tau \subseteq c \cap b\} = \{\tau \subseteq e\} \in \mathscr{F}^e \subseteq \mathscr{F}^b,$$

where

$$e = \mathrm{cl}(\mathrm{int}\, c \cap \mathrm{int}\, b) \in T.$$

We assume now that (8.3) holds. Then $D \in \mathscr{T}$ can be represented in the form

$$\Delta \equiv \{\tau \in D,\ a \subseteq \tau \subseteq b\} = \{\tau \subseteq b\} \cap \Gamma, \tag{8.6}$$

where Γ is some event from $\mathscr{F}^b$. For it is sufficient to prove (8.6) for sets D of the form $\{t\colon\ t \supseteq c\}$, $c \in T$, which, by Lemma 6.2(3), generate $\mathscr{T}$. But for such D if $\Delta \neq \emptyset$ we have $c \subseteq b$, hence

$$\Delta = \{\tau \supseteq c\} \cap \{\tau \supseteq a\} \cap \{\tau \subseteq b\},$$

where the first two events belong to $\mathscr{F}^b$, since $a \subseteq b$ and $c \subseteq b$. It remains to note that

$$\{\tau \nsubseteq b\} = \bigcup_j \{\tau \supseteq v_j\} \in \mathscr{F}^{\tilde{b}} \subseteq \mathscr{F}^{\tilde{a}},$$

where $\{v_j\}$ is the set of closed balls with rational centres and radii contained in $R^d \backslash b$. Thus, $\Delta = \Gamma \cap \Gamma'$, where $\Gamma \in \mathscr{F}^b$ and $\Gamma' \in \mathscr{F}^{\tilde{a}}$, as required.

8.5. We will not study domains of the form (8.2) and (8.3) in detail here; we give only some examples.

Let $\eta_x(\omega)$ be a real-valued field continuous in $x \in R^d$. Let $\eta_x(\omega)$ be adapted to the field ξ, that is, η_x is measurable with respect to $\mathscr{F}^t$ for all $t \in T$, $x \in t$. We assume that

$$\lambda(\omega) \equiv \{x\colon\ \eta_x(\omega) > 0\} \neq \emptyset,\quad \omega \in \Omega.$$

***Theorem* 8.5.** *The set* $\tau(\omega) = \mathrm{cl}\ \lambda(\omega)$ *has the property* (8.3). *If* $\tau(\omega)$ *is bounded for each* ω, *then it is a splitting random domain.*

Let $\zeta_x(\omega)$, $x \in R^d$, be another field with continuous realizations, adapted to ξ. We fix a point $x_0 \in R^d$ and a real number $q > 0$. We also assume that for all ω

$$x_0 \in \lambda(\omega), \quad \zeta_x(\omega) \geqslant 0.$$

We say that a point $x \in \lambda(\omega)$ belongs to the set $\lambda_q(\omega)$ if there is a (continuous) rectifiable curve $\gamma : I \equiv [0, 1] \to R^d$ such that $\gamma(0) = x_0$, $\gamma(1) = x$, $\gamma(r) \in \lambda(\omega)$ for $r \in I$, and

$$\int_\gamma \zeta_x(\omega)\, l\,(dx) < q,$$

where $l(dx)$ is the element of length. The set $\lambda_q(\omega)$ is non-empty and open; we denote its closure by $\tau_q(\omega)$. If, say, $\zeta_x(\omega) \geqslant \varkappa > 0$, then $\tau_q(\omega)$ is bounded and consequently $\tau_q(\omega) \in T$.

Let us suppose that the field $\xi_x(\omega), x \in \lambda(\omega)$, describes a non-homogenous medium and that $\zeta_x(\omega)^{-1}$ characterizes the local rate of propagation of some wave process at the point $x \in \lambda(\omega)$. Then according to the Huyghens principle the surface $\partial\tau_q(\omega)$ gives the position at time $q > 0$ of the front of the wave propagated in $\lambda(\omega)$ from x_0.

***Theorem* 8.6.** *The set $\tau_q(\omega)$ satisfies the condition* (8.2), *and if $\tau_q(\omega)$ is bounded, then it is a splitting random domain.*

Remark 8.2. If $\zeta_x \equiv 0$, then $\tau_q(\omega)$ is the closure of the connected component of an open set $\lambda(\omega)$ that contains x_0 (compare [3], [10], [11]).

The idea of the proof of Theorem 8.6 is close to that of Theorem 2 in [11].

8.6. We demonstrate one application of the random change of variables described in §5. Let T_2 be the set of all contours $t \subseteq R^2$ that can be represented as a union of a certain number of edges of the two-dimensional integer lattice $\mathbf{Z}^2$. If $t \in T_2$, we denote by $I(t)$ the (compact) domain bounded by the contour t. We put $t \leqslant s$ if $I(t) \subseteq I(s)$.

As before, let $(\Omega, \mathscr{F}, P)$ be a complete probability space. We assume that to each $x \in \mathbf{Z}^2$ there corresponds an automorphism $\theta_x : \Omega \to \Omega$ of the space $(\Omega, \mathscr{F}, P)$ such that $\theta_{x+y} = \theta_x \cdot \theta_y$. We consider a family $\mathscr{A}_i(a, b)$ $(i = 1, 2, 3;\ a \leqslant b)$ of σ-subalgebras of $\mathscr{F}$ that form a stochastic model on T_2. We say that this model is stationary if

$$\theta_x^{-1}\mathscr{A}_i(a, b) = \mathscr{A}_i(a+x, b+x),$$

where $a+x$ is the contour obtained from a by displacement through a vector x. A random function $\xi_t(\omega)$ taking values in some measurable space E_x for each $t \in T_2$ will be called stationary if $E_{t+x} = E_t$ and $\xi_{t+x}(\omega) = \xi_t(\theta_x\omega)$ (a.s.) for $t \in T_2$, $x \in \mathbf{Z}^2$.

Our goal is to show how by using Theorem 5.2 one can construct sufficiently broad classes of stationary Markov random functions and stationary Markov models on T_2. They are of interest as natural generalizations of Gibbs fields on $\mathbf{Z}^2$ and also in connection with the growing attention paid by physicists to fields defined not on points of space but on some other geometric objects (contours, loops, surfaces, "threads", and so on, see [16], [17]).

The raw material for the constructions will be random fields on $\mathbf{Z}^2$. Thus, the construction presented below can be regarded as a way of perturbing fields defined on points, which leads to fields defined on contours.

Let $\Xi_x(\omega)$, $x \in \mathbf{Z}^2$, be a random function taking values in a standard measurable space E such that

$$\Xi_{x+y}(\omega)=\Xi_y(\theta_x\omega), \qquad (x,\ y\in\mathbf{Z}^2,\ \omega\in\Omega).$$

For $v \subseteq R^2$ we denote by $\Xi_v(\omega)$ the configuration of values of $\Xi_x(\omega)$ on $v \cap \mathbf{Z}^2$. We consider the regular stochastic model $\mathfrak{A} = \{\mathcal{A}_t(a,\ b)\}$ generated by the random variable $t \mapsto \Xi_t(\omega)$, $t \in T_2$ (see § 1.2). This model is stationary. We also assume that the following condition holds.

8.A. The model $\mathfrak{A}$ is Markov.

For this it is sufficient, for example, that Ξ_x, $x \in \mathbf{Z}^2$, are independent. The condition 8.A is also fulfilled if Ξ_x is the Gibbs field with nearest neighbour potential [9].

Let $g : E \to [0, \infty)$ be a measurable function. We consider the functional

$$G(\omega,\ t)=\sum_{x\in t\cap\mathbf{Z}^2} g(\Xi_x(\omega)), \quad t\in T_2,$$

and for any $a \in T_2$ we put ${}^aZ = \{t \in T_2:\ t \geqslant a\}$.

Lemma 8.1. *The set ${}^aZ$ is a lattice. The functional $G(\omega,\ t)$ is submodular with respect to lattice operations in ${}^aZ$.*

Next, we assume that to each $a \in T_2$ there corresponds a (non-random) functional ${}^a\Phi(t)$, $t \in {}^aZ$, with the following properties.

8.B. For any $a \leqslant b,\ u \geqslant a,\ v \geqslant b$ the following inequality holds:

$${}^a\Phi(u\wedge v)+{}^b\Phi(u\vee v)\leqslant{}^a\Phi(u)+{}^b\Phi(v)$$

(the operations $\wedge$ and $\vee$ are understood in the sense of ${}^aZ$). For all $t \in T_2$ and $x \in \mathbf{Z}^2$ we have ${}^{a+x}\Phi(t+x) = {}^a\Phi(t)$.

8.C. For each $a \in T_2$ the set ${}^a\overline{Z}(\omega)$ of contours $t \in {}^aZ$ that minimize the functional

$${}^aF(\omega,\ t)={}^a\Phi(t)+G(\omega,\ t), \quad t\in{}^aZ,$$

is non-empty with probability 1.

We take an arbitrary map $c : T_2 \to \mathbf{Z}^2$ such that $c(t+x) = c(t)+x$ (for example, let $c(t)$ be the top right point of the contour t).

***Theorem* 8.7.** *For almost all* ω *the set* ${}^a\bar{Z}(\omega)$ *contains its infimum. Let* $\tau_a(\omega) = \inf {}^a\bar{Z}(\omega)$ *(a.s.),* $a \in T_2$. *Then* $\{\tau_a\}_{a\in T_2}$ *is a monotone family of weakly local splitting contours. The stochastic model*

$$\mathfrak{B} = \{\mathscr{B}_i(a, b)\} = \{\mathscr{A}_i(\tau_a, \tau_b)\} \quad (i = 1, 2, 3;\ a \leqslant b;\ a, b \in T_2)$$

is regular, stationary, and Markov. The random function $\zeta_t = (\Xi_{\tau_t}, \tau_t - c(t))$ *is stationary and Markov, and* $\sigma\{\zeta_t\} = \mathscr{B}_3(t, t)$.

We note that the spaces D_t of values of ζ_t are standard, hence they can be realized as Borel subsets of R^1, and ζ_t as a real-valued function.

Theorem 8.7 is a consequence of Theorem 5.2. The fact that ζ_t is Markov follows from the fact that the model $\mathfrak{B}$ is regular and Markov and the fact that $\sigma\{\zeta_t\} = \mathscr{B}_3(t, t)$.

We give examples of families of functionals ${}^a\Phi(\cdot)$, $a \in T_2$, satisfying the condition 8.B. Let $\Psi_j : R^1 \to R^1$ ($j = 1, \ldots, k$) be non-decreasing convex functions, and for each $j = 1, 2, \ldots, k$ let $\mu_j(t) = \max\{l_j(y), y \in t\}$, where l_j is some linear form on R^2. Let $\nu_j(a)$ be functions on T_2 such that $\nu_j(a+x) = \nu_j(a)$ for $a \in T_2$, $x \in \mathbf{Z}^2$ and $\nu_j(t) \geqslant \nu_j(s)$ if $t \geqslant s$. Then 8.B is satisfied for

$${}^a\Phi(t) = \sum_{j=1}^{k} \Psi_j(\mu_j(t) - \mu_j(a) - \nu_j(a)), \quad t \in {}^aZ,\ a \in T_2.$$

To guarantee the validity of condition 8.C we can introduce, for example, the assumption of sufficiently rapid growth of ${}^a\Phi(t)$ as the diameter of the contour t tends to infinity. Conditions of a different kind, which cover the case ${}^a\Phi(t) \equiv 0$, are the following: the functional ${}^a\Phi(t)$ is bounded from below in $t \in {}^aZ$; the random variables $\gamma_x = g(\Xi_x)$, $x \in \mathbf{Z}^2$, are independent and equally distributed, and $P\{\gamma_x > 0\} = 1$.

8.7. We consider some types of stochastic models different from $\mathfrak{A}(\xi)$ and connected with random fields ξ_x, $x \in R^d$. We put $T^d = R^d$, $\mathscr{T}^d = \mathscr{B}(R^d)$ and introduce on T^d a natural (coordinate-wise) relation $\leqslant$ of partial ordering of vectors. We denote by $\varphi(r)$ (respectively, $\psi(r)$) the integer that is nearest to r and smaller (respectively, larger) than r ($r \in R^1$), and put $\varphi_k(r) = 2^{-k}\varphi(2^k r)$, $\psi_k(r) = 2^{-k}\psi(2^k r)$. The family $\mathscr{H}^d$ of maps

$$f_k(t) = (\varphi_k(t_1), \ldots, \varphi_k(t_d)), \quad g_k(t) = (\psi_k(t_1), \ldots, \psi_k(t_d)),$$
$$t = (t_1, \ldots, t_d) \in R^d.$$

is a skeleton for $(T^d, \mathscr{T}^d, \leqslant)$.

Let $\mathfrak{G}(\xi) = \{\mathscr{G}_i(a, b)\}$ be the stochastic model on T^d generated by the random function ξ_x, $x \in T^d$ (see (1.1)) and let $\mathfrak{F}(\xi) = \{\mathscr{F}_i(a, b)\}$ be its closure (see § 1.3). It is clear that $\mathscr{F}_i(a, b)$ coincides with the intersection over $\varepsilon > 0$ of the σ-algebras

$$\mathscr{G}_i(a - \varepsilon e, b + \varepsilon e), \quad e = (1, 1, \ldots, 1) \in R^d.$$

We note that if $d = 1$, then the model $\mathfrak{S}(\xi)$ is Markov if and only if the stochastic process ξ_x, $x \in R^1$, is Markov (in the classical sense). At the same time, the Markov property of the model $\mathfrak{F}(\xi)$ for $d = 1$ is equivalent to the condition

$$\sigma\{\xi_{x'},\ x' \leqslant x\} \amalg \sigma\{\xi_{x'},\ x' \geqslant x\} \mid \bigcap_{\varepsilon>0} \sigma\{\xi_{x'},\ x' \in [x-\varepsilon,\ x+\varepsilon]\}.$$

Next, if $\xi_r^1, \ldots, \xi_r^d$ $(r \in R^1)$ are independent Markov processes, then the models $\mathfrak{S}(\xi)$ and $\mathfrak{F}(\xi)$ constructed from the random function $\xi_x = (\xi_{x_1}^1, \ldots, \xi_{x_d}^d)$ are Markov. Random functions of this kind and models of the type $\mathfrak{S}(\xi)$ arise, for example, in the study of harmonic functions and additive functionals of some Markov processes (see [43]). Markov functions of diffusion type on a partially ordered set of two-dimensional vectors were considered in [44].

We consider the model $\mathfrak{F}(\xi)$ in more detail in the one-dimensional case ($d = 1$). Here $\xi = \xi_x(\omega)$ $(\omega \in \Omega,\ \xi_x(\omega) \in E_x,\ x \in R^1)$ is a given stochastic process.

We define splitting moments τ as splitting random elements in the model $\mathfrak{F}(\xi)$ (see condition $(\mathscr{S})$ in §2.1). We say that the process ξ has the strong Markov property with respect to τ if the condition $(\mathscr{S}\mathscr{M})$ given in §2.1 is satisfied. We note that the σ-algebras $\mathscr{F}_i(\alpha, \beta)$ appearing in $(\mathscr{S}\mathscr{M})$ can be defined in an equivalent way as $\bigcap_{\varepsilon>0} \mathscr{F}_i^{(\varepsilon)}(\alpha, \beta)$, where $\mathscr{F}_i^{(\varepsilon)}(\alpha, \beta)$ are generated by α, β, and the events $\{\alpha - \varepsilon \leqslant a, \beta + \varepsilon \geqslant b\} \cap \Gamma$, $\Gamma \in \mathscr{F}_i(a, b)$ (this follows from Remark 6.2).

In the model $\mathfrak{F}(\xi)$ the requirement 2.A is fulfilled, since according to the general agreement (6.2) the process ξ_x, $x \in R^1$, generates (mod 0) the σ-algebra of all events $\mathscr{F}$. Thus, from Theorem 2.1 we deduce that the splitting condition $(\mathscr{S})$ and the strong Markov condition $(\mathscr{S}\mathscr{M})$ are equivalent in the model $\mathfrak{F}(\xi)$.

As a consequence of Theorem 4.2 we obtain the following result.

***Theorem* 8.8.** *Let η_x, $x \in R^1$, be a real stochastic process with continuous realizations such that for each x*

$$\eta_x \in \bigcap_{\varepsilon>0} \sigma\{\xi_y,\ x-\varepsilon \leqslant y \leqslant x+\varepsilon\}.$$

Let Z be a fixed compact subset of R^1 and τ the time when the process η_x for the first time attains its (absolute) minimum on Z. Then τ is a splitting random time.

8.8. We say a few words about one more important model. Let T^+ be the set of non-negative two-dimensional vectors with natural partial ordering $\leqslant$ and the Borel σ-algebra $\mathscr{T}^+$. Let ξ_x, $x \in T^+$, be a random function on T^+.

For $x = (x_1, x_2) \in T^+$ we put

$$M_1(x)=\{y\in T^+:\ 0\leqslant y_1\leqslant x_1\},\quad M_2(x)=\{y\in T^+:\ 0\leqslant y_2\leqslant x_2\},$$

$$M_3(x)=\{y\in T^+:\ 0\leqslant y\leqslant x\},\ y=(y_1,\ y_2);\quad M_i(a,\ b)=M_i(b);$$

$$\mathscr{Z}_i(a,\ b)=\sigma\{\xi_y,\ y\in M_i(a,\ b)\},\quad a\leqslant b,\quad a,\ b\in T^+.$$

The random functions ξ that are Markov in the sense of the model $\mathfrak{D} = \{\mathscr{Z}_i(a,\ b)\}$ (for example, the "Brownian sheet") play an important role in the theory of stochastic integrals on the plane, see [18].

8.9. We make some comments concerning the bibliography.

Various versions of the strong Markov property of random fields on R^d and on more general spaces have been studied in numerous papers [3], [10], [11], [23], [24], [45]-[48]. However, until recently the main attention has been paid to a version of this notion which is weaker than the one considered in this paper (it corresponds to the case $\alpha = \beta = \tau$ in the definition introduced in §6.4). In [10] and [11] a strong Markov property of this type was established for random domains satisfying condition (8.2). These random domains are direct analogues of stopping times (see [49]).(1)

Conditional probabilities of events connected with realizations of the field outside a random domain τ of the form (8.2) were also investigated in [11], [21]. It became clear that these conditional probabilities are given by the same law as if the domain τ were deterministic. This result generalizes the theorem which says that the evolution of a ("good" enough) Markov process after a stopping time τ is controlled by the same transition function as if the moment τ were non-random. This property is quite often included in the definition of the strong Markov property of a stochastic process; see, for example, [2]. We note that for splitting random domains of general form this result is not true. It can be seen on very simple examples (the one-dimensional case, a finite set of values of τ). The study of the strong Markov property of random domains more general than (8.2) was initiated in [23]. An analogue of this property for a fixed pair of domains $\alpha \subseteq \beta$ was investigated in [24]. Finally, the notion of the strong Markov property in the form given in this paper was introduced in [47], [48].

As regards applications of the strong Markov property of random fields, we mention [46], where results of the type of [11] were applied to asymptotic problems of mathematical statistics (limit theorems for empirical processes).

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(1)The analogues of stopping times for the partially ordered set $\{x \in R^2:\ x \geqslant 0\}$ were considered in [50], [51], and the references therein.

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